Thompson p-Nilpotence and Fusion Systems

David A. Craven

June 2011

1 Introduction

Thompson's normal p-complement theorem [7] [8] was one of the most important advances in the pre-classification era of finite group theory, and lead directly to Thompson's proof that a fixed-point-free automorphism of prime order only occurs for nilpotent (finite) groups.

As a p-complement theorem, Thompson's result can be expressed in the language of fusion systems (from which we can recover the theorem for groups by stating $\mathcal{F} = \mathcal{F}_P(G)$). The precise statement is as follows.

Theorem 1.1 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, where p is an odd prime. If $\mathrm{C}_{\mathcal{F}}(\mathrm{Z}(P))$ and $\mathrm{N}_{\mathcal{F}}(J(P))$ are trivial then $\mathcal{F}$ is trivial.

Here $J(P)$ is the *Thompson subgroup*; there are three different standard definitions of the Thompson subgroup, and we will prove this result for all of them.

Of course, Thompson's normal p-complement theorem has been proved for fusion systems before [4], but that method used both the original theorem for groups, and the fact that constrained saturated fusion systems come from finite groups, the constrained model theorem.

A standard application of Thompson's normal p-complement theorem is to prove that a finite group possessing a fixed-point-free automorphism of prime order is nilpotent. A *fixed-point-free* automorphism $\sigma \in \mathrm{Aut}(\mathcal{F})$ of a fusion system $\mathcal{F}$ is a fixed-point-free automorphism of the underlying subgroup P such that, if $Q \leqslant P$ is σ-invariant, then σ acts fixed-point-freely on $\mathrm{Aut}_{\mathcal{F}}(Q)$. An example of this is the action of a fixed-point-free automorphism of a finite group on its fusion system.

Theorem 1.2 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and suppose either that $\mathcal{F}$ is soluble or that p is odd. If $\mathcal{F}$ possesses a fixed-point-free automorphism of prime order then $\mathcal{F} = \mathcal{F}_P(P)$.

This theorem is enough to imply the corresponding result for finite groups.

Although a stronger theorem of this type [5] has been translated to fusion systems [6], the result passed via the finite group case. The purpose of this note is to eliminate recourse to the corresponding theorems for finite groups, and prove Theorems 1.1 and 1.2 directly for fusion systems, also bypassing the use of the constrained model theorem.

For an introduction to fusion systems, and for the definitions and concepts used in this paper, we refer the reader to, for example, [3].

2 The Thompson Subgroups

This short section recalls the three different standard definitions of the Thompson subgroup, $J(P)$, and will prove an easy lemma that holds regardless of which version of the Thompson subgroup one chooses.

Definition 2.1 Let G be a finite group. Define

$$\begin{aligned}\mathscr{J}_1(G) &= \{A \leqslant G : A \text{ is elementary abelian of maximal order in } G\},\\ \mathscr{J}_2(G) &= \{A \leqslant G : A \text{ is abelian of maximal order in } G\},\\ \mathscr{J}_3(G) &= \{A \leqslant G : A \text{ is abelian of maximal rank in } G\}.\end{aligned}$$

For each α, define $J_\alpha(G)$ to be the subgroup generated by all $A \in \mathscr{J}_\alpha(G)$. Write $J(G)$ for one of the $J_\alpha(G)$.

From this it is easy to see that $J(G)$ is characteristic in G, and non-trivial whenever G is non-trivial.

Lemma 2.2 Let P be a non-trivial finite p-group, and let A be an element of $\mathscr{J}_\alpha(P)$, for some α. If A_0 has index p in A, and V is any elementary abelian subgroup of P centralizing A_0, then $|A_0V : A_0| = |V : V \cap A_0| \leqslant p$.

Proof: Since V centralizes A_0, A_0V is abelian, and clearly $|A_0V : A_0| = |V : V \cap A_0|$. If $\alpha = 1$ or $\alpha = 2$ then $|A_0V| \leqslant |A|$, so that $|A_0V : A_0| \leqslant p$, as needed. If $\alpha = 3$, then since V is elementary abelian, $\Omega_1(A_0V) = \Omega_1(A_0)V$ and $V \cap A_0 = V \cap \Omega_1(A_0)$; in this case, $|\Omega_1(A_0V)|/|\Omega_1(A_0)| \leqslant p$, so we get $|V : V \cap A_0| \leqslant p$, completing the proof. $\square$

3 Basic Facts from Group Theory

Lemma 3.1 If G is a finite group with an abelian Sylow p-subgroup P, and $N = \mathrm{N}_G(P)$, then

$$P = \mathrm{C}_P(N) \times [P, N].$$

Furthermore, if H is a p'-group and $G = P \rtimes H$, then $P = \mathrm{C}_P(H) \times [P, H]$.

Lemma 3.2 If G is a subgroup of $\mathrm{GL}_2(p)$ with at least two Sylow p-subgroups then $G \geqslant \mathrm{SL}_2(p)$.

4 Working with Constrained Fusion Systems

Not using the constrained model theorem means that we have to work directly with constrained fusion systems. We therefore bring together a collection of useful results in this direction, doing more than we strictly need for the proof of Thompson's normal p-complement theorem because it might well be of independent interest.

We start with a useful lemma that follows easily from a fundamental result from Broto, Castellana, Grodal, Levi and Oliver, which restricts how many $\mathcal{F}$-conjugacy classes need to be check for saturation.

Lemma 4.1 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a normal subgroup of $\mathcal{F}$. If all $\mathcal{F}$-conjugacy classes of subgroups of P containing Q are saturated then $\mathcal{F}$ is saturated.

Proof: Notice that since Q is normal, $\mathcal{F}$ is generated by morphisms between subgroups of P containing Q. By [1, Theorem 2.2], $\mathcal{F}$ is saturated if, for all $\mathcal{F}$-conjugacy classes $\mathcal{R}$ of $\mathcal{F}$-centric subgroups of P not containing Q, there is an element $R \in \mathcal{R}$ such that $\mathrm{Out}_P(R) \cap \mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(R))$ is non-trivial.

Consider $\mathrm{Aut}_{QR}(R)$, which we claim is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$. Let $x \in \mathrm{N}_{QR}(R)$ and let $\phi \in \mathrm{Aut}_{\mathcal{F}}(R)$, with $\psi \in \mathrm{Aut}_{\mathcal{F}}(QR)$ an extension to QR. Since R and Q are ψ-invariant, $\mathrm{N}_{QR}(R)$ is ψ-invariant. Hence

$$(c_x)^{\psi} = c_{x\psi} \in \mathrm{Aut}_{QR}(R).$$

Thus $\mathrm{Aut}_{QR}(R)$ is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$. If $\mathrm{Out}_P(R) \cap \mathrm{O}_p(\mathrm{Out}_{\mathcal{F}}(R)) = 1$ then $\mathrm{Aut}_{QR}(R) = \mathrm{Inn}(R)$, and so $QR \leqslant R\,\mathrm{C}_P(R) = R$. Hence $Q \leqslant R$, a contradiction. This completes the proof. □

Using Lemma 4.1, we can construct many saturated subsystems of a constrained fusion system.

Lemma 4.2 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a normal, centric subgroup of P. If H is a subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$ such that $\mathrm{Out}_P(Q) \cap H$ is a Sylow p-subgroup of H, then there exists a unique saturated subsystem $\mathcal{E}$ of $\mathcal{F}$, on a subgroup of P containing Q, such that $\mathrm{Out}_{\mathcal{E}}(Q) = H$.

Proof: Let R be the preimage in P of $\mathrm{Out}_P(Q) \cap H$, which contains Q. For $Q \leqslant A, B \leqslant R$, define $\mathrm{Hom}_{\mathcal{E}}(A, B)$ to be the subset of all $\phi \in \mathrm{Hom}_{\mathcal{F}}(A, B)$ such that $\phi|_Q \in \bar{H}$, where $\bar{H}$ is the preimage of H in $\mathrm{Aut}_{\mathcal{F}}(Q)$. For general A, B we define $\mathrm{Hom}_{\mathcal{E}}(A, B)$ to be the restriction to A of all $\phi \in \mathrm{Hom}_{\mathcal{E}}(AQ, BQ)$ such that $A\phi \leqslant B$. It is easy to see that $\mathcal{E}$ is a subsystem of $\mathcal{F}$; we claim that $\mathcal{E}$ is saturated. Clearly Q is normal in $\mathcal{E}$.

By Lemma 4.1 it suffices to show that the $\mathcal{E}$-conjugacy classes of subgroups of R containing Q are saturated. Let $Q \leqslant B \leqslant R$, and we first note that B is receptive in $\mathcal{E}$. To see this, let $\phi : A \to B$ be an $\mathcal{E}$-isomorphism: since Q is $\mathcal{F}$-centric, A and B are $\mathcal{F}$-centric, so receptive in $\mathcal{F}$. Let N_ϕ^R denote the preimage in R of $\mathrm{Aut}_R(A) \cap \mathrm{Aut}_R(B)^{\phi^{-1}}$, which is contained in N_ϕ^P, which is the standard subgroup N_ϕ with respect to $\mathcal{F}$. As B is $\mathcal{F}$-receptive, ϕ extends to $\bar{\phi} : N_\phi^P \to \mathrm{N}_P(B)$, and this restricts to $\psi : N_\phi^Q \to \mathrm{N}_P(B)$, which lies in $\mathcal{E}$. (It lies in $\mathcal{E}$ since it restricts to an automorphism of Q lying in $\bar{H}$, namely $\phi|_Q$; in particular, we see that $\operatorname{im}\psi \leqslant R$.) Hence B is receptive in $\mathcal{E}$, as needed.

It remains to prove that A is $\mathcal{E}$-conjugate to a fully $\mathcal{E}$-automized subgroup of R. Let $\kappa : \mathrm{Aut}_\mathcal{F}(A) \to \mathrm{Aut}_\mathcal{F}(Q)$ be the restriction map: the kernel $\ker\kappa$ consists of those automorphisms of A acting trivially on the $\mathcal{F}$-centric subgroup Q, and so by [2, Proposition A.8] consists of maps c_g for $g \in \mathrm{Z}(Q)$ (with $c_g = c_h$ if and only if $gh^{-1} \in \mathrm{Z}(R)$, in particular $|\ker\kappa| = |\mathrm{Z}(Q)/\mathrm{Z}(R)|$, and $\ker\kappa \leqslant \mathrm{Inn}(R)$).

Since $\ker\kappa \leqslant \mathrm{Inn}(A)$, we see that $\mathrm{Aut}_R(A)$ is a Sylow p-subgroup of $\mathrm{Aut}_\mathcal{E}(A)$ if and only if $\mathrm{Aut}_R(A)\kappa$ is a Sylow p-subgroup of $\mathrm{Aut}_\mathcal{E}(A)\kappa$, which is equal to $\mathrm{N}_{\bar{H}}(\mathrm{Aut}_A(Q))$. Let $\chi \in \bar{H}$ be such that a Sylow p-subgroup of $[\mathrm{N}_{\bar{H}}(\mathrm{Aut}_A(Q))]^\chi$ is contained in $\mathrm{Aut}_R(Q)$, which can be done since $\mathrm{Aut}_R(Q) = \mathrm{Aut}_P(Q) \cap \bar{H}$ is a Sylow p-subgroup of $\bar{H}$ by assumption.

Write B for the preimage in R of $\mathrm{Aut}_A(Q)^\chi$, so that in particular $Q \leqslant B$ and $\mathrm{Aut}_A(Q)^\chi = \mathrm{Aut}_B(Q)$. We claim that B is fully $\mathcal{E}$-automized. By the previous paragraph, it suffices to show that $\mathrm{Aut}_R(B)\kappa$ is a Sylow p-subgroup of $\mathrm{N}_{\bar{H}}(\mathrm{Aut}_B(Q))$. However, $\mathrm{Aut}_R(B)\kappa = \mathrm{Aut}_{\mathrm{N}_R(B)}(Q)$, and $\mathrm{Aut}_R(Q) \cap \mathrm{N}_{\bar{H}}(\mathrm{Aut}_B(Q)) = \mathrm{N}_{\mathrm{Aut}_R(Q)}(\mathrm{Aut}_B(Q)) = \mathrm{Aut}_{\mathrm{N}_R(B)}(Q)$. This proves that B is fully $\mathcal{E}$-automized, and concludes the proof of existence.

To prove uniqueness, suppose that $\mathcal{E}'$ is another saturated subsystem of $\mathcal{F}$ such that $\mathrm{Out}_{\mathcal{E}'}(Q) = H$. Again, $\mathcal{E}'$ is defined on R, so by Alperin's fusion theorem it suffices to show that $\mathrm{Aut}_\mathcal{E}(A) = \mathrm{Aut}_{\mathcal{E}'}(A)$ for every fully normalized subgroup $A \geqslant Q$. To see this, [2, Proposition A.8] proves that the kernels of the restriction maps $\mathrm{Aut}_\mathcal{F}(A) \to \mathrm{Aut}_\mathcal{F}(Q)$, $\mathrm{Aut}_\mathcal{E}(A) \to \mathrm{Aut}_\mathcal{E}(Q)$ and $\mathrm{Aut}_{\mathcal{E}'}(A) \to \mathrm{Aut}_{\mathcal{E}'}(Q)$ have order $|\mathrm{Z}(Q)|/|\mathrm{Z}(A)|$ and lie in $\mathrm{Inn}(A)$, hence are equal. As the images of $\mathrm{Aut}_\mathcal{E}(A)$ and $\mathrm{Aut}_{\mathcal{E}'}(A)$ are the same, namely $\mathrm{N}_{\bar{H}}(\mathrm{Aut}_A(Q))$, we see that $\mathrm{Aut}_\mathcal{E}(A) = \mathrm{Aut}_{\mathcal{E}'}(A)$, and the result follows. □

This yields the following corollaries.

Corollary 4.3 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and suppose that $\mathcal{F}$ contains a normal, $\mathcal{F}$-centric subgroup Q. Write $G = \mathrm{Out}_\mathcal{F}(Q)$. There is a one-to-one correspondence between subgroups H of G for which $H \cap \mathrm{Out}_P(Q)$ is a Sylow p-subgroup of H, and saturated subsystems $\mathcal{E}$ of $\mathcal{F}$ on subgroups of P containing Q, given by sending $\mathcal{E}$ to $\mathrm{Out}_\mathcal{E}(Q)$.

Proof: The only thing left to remark, beyond Lemma 4.2, is that $\mathrm{Out}_\mathcal{E}(Q) \cap \mathrm{Out}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Out}_\mathcal{E}(Q)$, since $\mathcal{E}$ is saturated and Q is strongly $\mathcal{E}$-closed, hence fully $\mathcal{E}$-normalized. □

Corollary 4.4 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and suppose that $\mathcal{F}$ contains a normal, $\mathcal{F}$-centric subgroup Q. If $\mathcal{F}/Q$ is trivial then $\mathrm{Out}_\mathcal{F}(Q)$ contains a normal p-complement.

Proof: Suppose that $G = \mathrm{Out}_{\mathcal{F}}(Q)$ does not have a normal p-complement, so that there exists a non-trivial p-subgroup X of G such that $\mathrm{N}_G(X)/\mathrm{C}_G(X)$ is not a p-group, by Frobenius's normal p-complement theorem. Let $\mathcal{E}$ be the saturated subsystem of $\mathcal{F}$ corresponding to $\mathrm{N}_G(X)$. We claim that $\mathcal{E}/Q$ is non-trivial, and in particular if R is the preimage of X in P, then $\mathrm{N}_{\mathcal{E}}(R)/Q$ is non-trivial.

To see this, notice that $\mathrm{Out}_{\mathcal{E}}(Q) = \mathrm{N}_{\mathrm{Out}_{\mathcal{F}}(Q)}(X)$, so that $\mathrm{Aut}_{\mathcal{E}}(Q) = \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_R(Q))$, and every $\mathcal{E}$-automorphism of Q extends to an $\mathcal{E}$-automorphism of R. Consider the map $\kappa : \mathrm{Aut}_{\mathcal{E}}(R) \to \mathrm{Aut}_{\mathcal{E}/Q}(R/Q)$. The kernel $\ker \kappa$ consists of those $\phi \in \mathrm{Aut}_{\mathcal{E}}(R)$ such that ϕ acts trivially on R/Q, or equivalently – since Q is $\mathcal{E}$-centric – $\phi|_Q$ acts trivially on $\mathrm{Out}_R(Q) = X$. This means that $\phi|_Q$ (modulo $\mathrm{Inn}(Q)$) lies in $\mathrm{C}_G(X)$. However, by assumption $\mathrm{N}_G(X)/\mathrm{C}_G(X)$ is *not* a p-group, so $\mathrm{Aut}_{\mathcal{E}/Q}(R/Q)$ is not a p-group, as needed. $\square$

5 Thompson's Normal p-Complement Theorem

In this section we give a proof of Theorem 1.1. We let the Thompson subgroup be $J_\alpha(G)$ for any α. Let $\mathcal{F}$ be a minimal counterexample firstly in terms of $|P|$, then in terms of the number of morphisms in $\mathcal{F}$.

Step 1: *$\mathcal{F}$ is sparse, $\mathrm{N}_{\mathcal{F}}(P) = \mathcal{F}_P(P)$, and if $\mathrm{Z}(P) \leqslant X \leqslant P$ and $\mathcal{E} \leqslant \mathcal{F}$ is defined on a overgroup of X in which X is fully $\mathcal{E}$-centralized, then $\mathrm{C}_{\mathcal{E}}(X)$ is trivial.* The first part is true since any proper subsystem of $\mathcal{F}$ on P satisfies the conditions of the result, so is $\mathcal{F}_P(P)$; the second part is true since else $P \trianglelefteqslant \mathcal{F}$ and so $J(P) \trianglelefteqslant \mathcal{F}$, a contradiction; the third part is clear.

Step 2: *If $Q = \mathrm{O}_p(\mathcal{F})$ then $\mathcal{F}/Q$ is trivial.* We first prove that $\mathrm{O}_p(\mathcal{F}) > 1$. By Alperin's fusion theorem, since $\mathcal{F}$ is non-trivial, there is some non-trivial subsystem $\mathrm{N}_{\mathcal{F}}(Q)$ for some $Q \leqslant P$. Choose Q so that $N = \mathrm{N}_P(Q)$ has maximal order; we claim that $N = P$, so assume false.

By induction, writing $\mathcal{N} = \mathrm{N}\,\mathcal{F}(Q)$, since $\mathcal{N}$ is non-trivial either $\mathrm{N}_{\mathcal{N}}(J(N))$ or $\mathrm{C}_{\mathcal{N}}(\mathrm{Z}(N))$ is non-trivial. Since $\mathrm{Z}(P) \leqslant \mathrm{Z}(N)$, $\mathrm{C}_{\mathcal{N}}(\mathrm{Z}(N))$ is trivial by Step 1, so $\mathrm{N}_{\mathcal{N}}(J(N))$ must be non-trivial. However, as $J(N) \,\mathrm{char}\, N \trianglelefteqslant \mathrm{N}_P(N)$, we see that $\mathrm{N}_P(J(N)) > N$. By choice of Q, $\mathrm{N}_{\mathcal{F}}(J(N))$ is therefore trivial, so clearly $\mathrm{N}_{\mathcal{N}}(J(N))$ is also trivial. This contradiction proves that $N = P$, and since $\mathrm{N}_{\mathcal{F}}(Q)$ is non-trivial, $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$. Set $Q = \mathrm{O}_p(\mathcal{F})$.

If $Q < W \trianglelefteqslant P$, then $\mathrm{N}_{\mathcal{F}}(W) < \mathcal{F}$, so $\mathrm{N}_{\mathcal{F}}(W)$ is trivial. Write $\bar{\ }$ for quotienting by Q. Since P cannot be normal in $\mathcal{F}$, $\bar{P} \neq 1$. Notice that if W is the preimage of $J(\bar{P})$ or $\mathrm{Z}(\bar{P})$ in P, then $Q < W \trianglelefteqslant P$, and so $\mathrm{N}_{\mathcal{F}}(W)$ is trivial. Thus $\mathrm{N}_{\bar{\mathcal{F}}}(\bar{W}) = \mathrm{N}_{\mathcal{F}}(W)/Q$ is also trivial, and therefore so is $\mathrm{C}_{\bar{\mathcal{F}}}(\bar{W})$. In particular, $\mathrm{N}_{\bar{\mathcal{F}}}(J(\bar{P}))$ and $\mathrm{C}_{\bar{\mathcal{F}}}(\mathrm{Z}(\bar{P}))$ are trivial, so by induction $\bar{\mathcal{F}}$ is trivial, as needed.

Step 3: *$\mathcal{F}$ is constrained.* Stancu's lemma states that $\mathcal{F} = \langle P\,\mathrm{C}_{\mathcal{F}}(Q), \mathrm{N}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))\rangle$. By Step 1, both of these are either $\mathcal{F}_P(P)$ or $\mathcal{F}$. If $\mathcal{F} = P\,\mathrm{C}_{\mathcal{F}}(Q)$ then by [3, Proposition 5.60] there is a bijection between essential subgroups of $\mathcal{F}$ and those of $\mathcal{F}/\mathrm{O}_p(\mathcal{F})$, and since the latter is trivial the former is. Hence we must have $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))$, and so $\mathcal{F}$ is constrained.

Step 4: $\mathrm{Out}_{\mathcal{F}}(Q) = H \rtimes \mathrm{Out}_P(Q)$, *where* H *is an elementary abelian* q*-group for some* $q \neq p$. *Furthermore,* $\mathrm{Out}_P(Q)$ *is maximal in* $\mathrm{Out}_{\mathcal{F}}(Q)$, *and every non-trivial normal subgroup of* $\mathrm{Out}_{\mathcal{F}}(Q)$ *contains* H. Write $G = \mathrm{Out}_{\mathcal{F}}(Q)$: by Step 2 and Corollary 4.4, $G = H \rtimes \mathrm{Out}_P(Q)$ for some p'-group H; since $\mathcal{F}$ is sparse, $\mathrm{Out}_P(Q)$ is maximal in G by Corollary 4.3, so that any Sylow p-subgroup of G is maximal in G. Since $\mathrm{Out}_P(Q)$ is maximal in G, $G = K\,\mathrm{Out}_P(G)$ for any normal p'-subgroup K of G, and hence G possesses a single normal p'-subgroup, which must be H.

Let $q \neq p$ be a prime dividing $|H|$, let R be a Sylow q-subgroup of H and $R_0 = \Omega_1(\mathrm{Z}(R))$. By the Frattini argument, $G = \mathrm{N}_G(R)H$, so $\mathrm{N}_G(R)$ contains a Sylow p-subgroup of G (which is maximal); $\mathrm{N}_G(R)$ also contains R, so $\mathrm{N}_G(R) = G$, and $R_0 \operatorname{char} R \trianglelefteqslant G$, proving that $R_0 \trianglelefteqslant G$. Thus $H = R_0$ is elementary abelian. Finally, if K is a normal subgroup not contained in H then $K \cap H = 1$, so that K is a normal p-subgroup of G. The preimage of a normal p-subgroup of G in P is a normal subgroup of $\mathcal{F}$, so $K = 1$ as $Q = \mathrm{O}_p(\mathcal{F})$; hence $H \leqslant K$.

Since $\mathrm{N}_{\mathcal{F}}(J(P))$ is trivial, we cannot have that $J(P) \leqslant Q$, since else $J(P) = J(Q) \operatorname{char} Q$, and so $J(P) \trianglelefteqslant \mathcal{F}$. Hence there exists some abelian p-subgroup A in $\mathscr{J}_\alpha(P)$ such that $A \nleqslant Q$. We may choose A to be fully $\mathcal{F}$-normalized.

Step 5: $P = QA$ *and* $P/Q \cong A/(A \cap Q)$ *has order* p. Write B for the image of A in $\mathrm{Out}_P(Q)$. Since $H \trianglelefteqslant \mathrm{Out}_{\mathcal{F}}(Q)$, $[H, B]$ is a subgroup of H normalized by B, so is normal in HB (as H is abelian, so normalizes $[H, B]$). Let H_1 be a minimal normal subgroup of HB contained in $[H, B]$. Let $\mathcal{E}$ be the saturated subsystem of $\mathcal{F}$ such that $\mathrm{Out}_{\mathcal{E}}(Q) = H_1B$, which exists by Lemma 4.2. Clearly $\mathcal{E}$ is non-trivial, and $\mathcal{E}$ is defined on QA. By Step 1, $\mathrm{C}_{\mathcal{E}}(\mathrm{Z}(QA))$ is trivial. If $\mathrm{N}_{\mathcal{E}}(J(QA))$ is non-trivial then there exists a non-trivial p'-automorphism ϕ in it: since both Q and $J(QA)$ are normal subgroups of $\mathrm{N}_{\mathcal{E}}(J(QA))$, so is $QJ(QA) = QA$, and hence ϕ extends to an automorphism $\psi \in \mathrm{Aut}_{\mathrm{N}_{\mathcal{E}}(J(QA))}(QA)$. As ψ is not a p-automorphism, by raising to an appropriate power, we may assume that ψ is a p'-automorphism of QA in $\mathcal{E}$.

As ψ is a p'-automorphism, if $\psi|_Q$ is trivial, then since $\mathrm{C}_P(Q) \leqslant Q$, ψ is trivial; thus $\psi|_Q$ is non-trivial. We consider the image of ψ in $\mathrm{Out}_P(Q)$. Since ψ normalizes QA, $[B, \psi] \leqslant B$, and since $\psi \in H$ we have that $[B, \psi] \leqslant H$, so that $[B, \psi] = 1$. Hence $\psi \in \mathrm{C}_H(B)$; as $\psi \in [H, B]$ we get that $\psi \in [H, B] \cap \mathrm{C}_H(B) = 1$ (by Lemma 3.1). Thus $\mathrm{N}_{\mathcal{E}}(J(QA))$ is trivial, and so by choice of minimal counterexample, $\mathcal{E} = \mathcal{F}$. In particular, $P = QA$.

Notice that H is a simple, faithful, $\mathbb{F}_q P/Q$-module, and since P/Q is abelian we must have that P/Q is cyclic. Thus $A/(A \cap Q)$ has order p.

Write $Z = \mathrm{Z}(Q)$. Since $\mathrm{Z}(P) \leqslant Z$ (as $\mathrm{C}_P(Q) \leqslant Q$) we see by Step 1 that $\mathrm{C}_{\mathcal{F}}(Z)$ is a trivial fusion system on $\mathrm{C}_P(Z)$. By Lemma 3.1, $Z = \mathrm{C}_Z(H) \times [Z, H]$, and so if $[Z, H] = 1$ then H centralizes Z, impossible since $\mathrm{C}_{\mathcal{F}}(Z) = \mathcal{F}_P(P)$. Hence $[H, Z] \neq 1$. Let $V = \Omega_1([H, Z])$, an elementary abelian p-group. Also, as $Q \trianglelefteqslant \mathcal{F}$, $Z \trianglelefteqslant \mathcal{F}$, and so $[H, Z] \trianglelefteqslant \mathcal{F}$, and $\Omega_1([H, Z]) \trianglelefteqslant \mathcal{F}$.

Step 6: $|V| \leqslant p^2$. Let $A_0 = A \cap Q$; as AQ/Q has order p, A/A_0 has order p. As $V \leqslant \mathrm{Z}(Q)$, V centralizes A_0, so by Lemma 2.2 we have $|V/(V \cap A_0)| \leqslant p$. If $\phi \in \mathrm{Aut}_{\mathcal{F}}(Q)$ then $V\phi = V$, and so

$V/(V \cap A_0)\phi = V/(V \cap (A_0\phi))$ has order at most p. If $|V| \geqslant p^3$ then $X = V \cap A_0 \cap A_0\phi \neq 1$. As A is abelian, $\mathrm{Aut}_A(Q)$ acts trivially on X, as does $\mathrm{Inn}(Q)$ since $X \leqslant \mathrm{Z}(Q)$. As $\mathrm{Aut}_A(Q)$ centralizes A_0, $\mathrm{Aut}_A(Q)^\phi$ centralizes $A_0\phi \geqslant X$, and so X is centralized by $\langle \mathrm{Aut}_A(Q), \mathrm{Aut}_A(Q)^\phi, \mathrm{Inn}(Q)\rangle = \mathrm{Aut}_{\mathcal{F}}(Q)$. Hence $X \leqslant \mathrm{C}_H(Z)$, which as $X \leqslant [H,Z]$ gives $X \leqslant \mathrm{C}_H(Z) \cap [H,Z] = 1$, a contradiction. Thus $|V| \leqslant p^2$.

Step 7: $\mathrm{Out}_{\mathcal{F}}(Q) \geqslant \mathrm{SL}_2(p)$ *and contradiction.* If we can show that $C = \mathrm{C}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(V) = \mathrm{Inn}(Q)$ then $\mathrm{Out}_{\mathcal{F}}(Q)$ may be embedded in $\mathrm{Aut}(V) = \mathrm{GL}_2(p)$. By Step 4, $\mathrm{Out}_{\mathcal{F}}(Q)$ has at least two Sylow p-subgroups, so $\mathrm{Out}_{\mathcal{F}}(Q) \geqslant \mathrm{SL}_2(p)$ by Lemma 3.2; this contradicts Step 4, as $\mathrm{Out}_{\mathcal{F}}(Q) = H \rtimes \mathrm{Out}_P(Q)$.

Since V is normalized by $\mathrm{Aut}_{\mathcal{F}}(Q)$, $C \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$, whence $C/\mathrm{Inn}(Q)$ is a normal subgroup of $\mathrm{Out}_{\mathcal{F}}(Q)$. By Step 2, if $C > \mathrm{Inn}(Q)$ then it contains H, so that $V \leqslant \mathrm{C}_Z(H) \cap [H,Z] = 1$; hence $C = \mathrm{Inn}(Q)$, as claimed. This completes the proof of Theorem 1.1.

If we allow $p = 2$ and require that $\mathcal{F}$ is S_3-free – i.e., S_3 is not involved in $\mathrm{Aut}_{\mathcal{F}}(Q)$ for any subgroup Q of P – then $\mathrm{Aut}_{\mathcal{F}}(Q) \not\geqslant \mathrm{SL}_2(2) = S_3$ in Step 7 and the theorem also holds in this case. If $\mathcal{F} = \mathcal{F}_P(G)$, where P is a Sylow 2-subgroup of G, then G is S_4-free if and only if $\mathcal{F}$ is S_3-free, this naturally extends Thompson's normal p-complement theorem to the prime 2. (As the fusion system of S_4 itself satisfies $\mathrm{N}_{\mathcal{F}}(J(P)) = \mathrm{C}_{\mathcal{F}}(\mathrm{Z}(P))) = \mathcal{F}_P(P)$ but $\mathrm{O}_{2'}(S_4) = 1$, we do need to exclude some finite groups for the prime 2.)

6 Fixed-Point-Free Automorphisms of Groups

If G is a finite group then an automorphism σ of G is *fixed point free* if $g\sigma = g$ implies $g = 1$. In this section we produce a few preliminary lemmas on fixed-point-free automorphisms of finite groups, needed for the proofs in the next section. All of these results are well known.

Lemma 6.1 Let G be a finite group, and let ϕ be a fixed-point-free automorphism of order n. We have that every element of G can be written in the form $x^{-1}(x\phi)$ or $(x\phi)x^{-1}$, and for all $x \in G$,

$$x(x^\phi)(x^{\phi^2})\dots(x^{\phi^{n-1}}) = 1.$$

Proof: If $x^{-1}x^\phi = y^{-1}y^\phi$ then $yx^{-1} = (y\phi)(x\phi)^{-1} = (yx^{-1})\phi$, so that $y = x$. Hence the map $x \mapsto x^{-1}x^\phi$ is an injection, so is a bijection as $|G|$ is finite: this proves the first part.

To see the second, write $x = y^{-1}y^\phi$: then

$$x(x^\phi)(x^{\phi^2})\dots(x^{\phi^{n-1}}) = (y^{-1}y^\phi)(y^{-1}y^\phi)^\phi \dots (y^{-1}y^\phi)^{\phi^{n-1}} = y^{-1}y^{\phi^n} = 1,$$

as the middle terms cancel each other off, and ϕ has order n. □

Lemma 6.2 Let G be a finite group, and let ϕ be a fixed-point-free automorphism of G. If p is a prime dividing $|G|$, then ϕ fixes a unique Sylow p-subgroup P of G.

Proof: If P is a Sylow p-subgroup of G, then $P\phi$ is also a Sylow p-subgroup of G. Therefore, $P\phi = x^{-1}Px$ for some $x \in G$. Thus for any $y \in G$,

$$(y^{-1}Py)\phi = (y^{\phi})^{-1}x^{-1}Px(y^{\phi}).$$

However, every element of G can be expressed as $x^{\phi}x^{-1}$ by Lemma 6.1 and so choose y such that $y^{\phi}y^{-1} = x^{-1}$; then P^y is fixed under ϕ, as required.

Now suppose that P and P^x are fixed by ϕ. Therefore, $x^{\phi}x^{-1} \in \mathrm{N}_G(P)$. Again, since $x^{\phi}x^{-1}$ is in $\mathrm{N}_G(P)$, we see that (as $\mathrm{N}_G(P)$ is ϕ-invariant) there is an element $y \in \mathrm{N}_G(P)$ such that

$$x^{\phi}x^{-1} = y^{\phi}y^{-1}.$$

Since the map $x \mapsto x^{\phi}x^{-1}$ is a bijection, $x = y$, and so $P^x = P$, as needed. $\square$

Lemma 6.3 Let G be a finite abelian group, and suppose that H is a subgroup of $\mathrm{Aut}(G)$ of the form $K \rtimes \langle\phi\rangle$. Suppose that, for all $k \in K$, the element $k\phi$ is fixed point free and of prime order p, and that $|K|$ and $|G|$ are coprime. Then K fixes some non-trivial element of G.

Proof: For each $k \in K$, and $x \in G$ the element

$$x^{1+(k\phi)+(k\phi)^2+\cdots+(k\phi)^{p-1}} = 1,$$

by Lemma 6.1. If $x \in G$, then we can multiply these together for all $k \in K$, and have

$$1 = \prod_{k\in K}\prod_{i=0}^{p-1} x^{(k\phi)^i} = x^{|K|}\prod_{i=1}^{p-1}\prod_{k\in K} x^{(k\phi)^i}.$$

Notice that clearly $\{k\phi^i \,:\, k \in K,\, 1 \leqslant i \leqslant p-1\} = H \setminus K$, but we claim also that

$$\{(k\phi)^i \,:\, k \in K,\, 1 \leqslant i \leqslant p-1\} = H \setminus K.$$

To see this, suppose that $(k\phi)^i = (l\phi)^j$, for some $1 \leqslant i, j \leqslant p-1$ and $k, l \in K$. The image of these maps in H/K is $K\phi^i$ and $K\phi^j$, so that $i = j$. Since $k\phi$ has order p, by raising to a certain power i' such that $ii' \equiv 1 \pmod p$, we get $(k\phi)^{ii'} = k\phi = l\phi = (l\phi)^{ii'}$, so that clearly $k = l$.

Applying this to the product above, we get that

$$1 = x^{|K|}\prod_{i=1}^{p-1}\prod_{k\in K} x^{(k\phi)^i} = x^{|K|}\prod_{i=1}^{p-1}\prod_{k\in K} x^{k\phi^i}.$$

Clearly, $\prod_{k\in K} x^{k\phi^i}$ is a fixed point of G under the action of all $k \in K$, and since $x^{|K|}$ is not the identity, one of the terms in the product must also not be the identity. Hence there is a fixed point of G under the action of K. $\square$

7 Fixed-Point-Free Automorphisms of Fusion Systems

For a fusion system $\mathcal{F}$ on a finite p-group P, an automorphism $\sigma \in \mathrm{Aut}(\mathcal{F})$ is *fixed point free* if σ acts fixed point freely on P, and whenever $Q \leqslant P$ is σ-invariant, the induced action of σ on $\mathrm{Aut}_{\mathcal{F}}(Q)$ is fixed point free.

A fundamental theorem of Thompson on fixed-point-free automorphisms of finite groups is that if G possesses a fixed-point-free automorphism of prime order then G is nilpotent. For soluble groups this is proved directly, but for arbitrary groups this uses a p-nilpotence theorem like that described above.

Proposition 7.1 Let $\mathcal{F}$ be a soluble, saturated fusion system on a finite p-group P. If $\mathcal{F}$ possesses a fixed-point-free automorphism σ of prime order then $\mathcal{F} = \mathcal{F}_P(P)$.

Proof: Let $\mathcal{F}$ be a minimal counterexample to the statement. Since $\mathcal{F}$ is trivial if and only if $\mathcal{F}/\mathrm{Z}(\mathcal{F})$ is, and $\mathrm{Z}(\mathcal{F})$ is characteristic in $\mathcal{F}$, we may assume by induction that $\mathrm{Z}(\mathcal{F}) = 1$. Let Q be a minimal σ-invariant normal subgroup of $\mathcal{F}$, which exists since $\mathrm{O}_p(\mathcal{F}) > 1$, and is clearly elementary abelian. Since Q is σ-invariant, σ acts fixed point freely on $\mathrm{Aut}_{\mathcal{F}}(Q)$. Since $\mathrm{Z}(\mathcal{F}) = 1$, $\mathcal{F} \neq P\,\mathrm{C}_{\mathcal{F}}(Q)$ by the hypercentral subgroup theorem, so that $\mathrm{Aut}_{\mathcal{F}}(Q)$ is not a p-group. Let R be the σ-invariant Sylow r-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ for some $r \neq p$ dividing $|\,\mathrm{Aut}_{\mathcal{F}}(Q)|$.

It is clear that there is a saturated subsystem $\mathcal{E}$ of $\mathcal{F}$ on Q such that $\mathrm{Aut}_{\mathcal{E}}(Q) = R$. (Simply take the fusion system of $\mathcal{F}_Q(Q \rtimes R)$.) We have that $\mathcal{E}^\sigma = \mathcal{E}$, so by choice of minimal counterexample $\mathcal{F} = \mathcal{E}$.

We aim to apply Lemma 6.3 to this situation: Since both R and σ act on Q, and σ acts on R, writing $K = \mathrm{Aut}_R(Q)$, we have $H = K \rtimes \langle\sigma\rangle$ as a subgroup of $\mathrm{Aut}(Q)$. We must show that, for each $k \in K$, $k\sigma$ has order p and acts fied point freely, which we will accomplish by showing it is conjugate to σ. Then $K = \mathrm{Aut}_R(Q)$ fixes a point $1 \neq x \in Q$, and so $x \in \mathrm{Z}(\mathcal{F})$, a contradiction to the fact that $\mathrm{Z}(\mathcal{F}) = 1$.

To show that $\sigma^\alpha = k\sigma$ for some $\alpha \in H$, let $\ell \in K$ be such that $\ell^{-1}(\ell^{\sigma^{-1}}) = k$, which exists by Lemma 6.1. Notice that $\ell k = \sigma\ell\sigma^{-1}$, so that $k\sigma = \ell^{-1}\sigma\ell$, as needed. □

Theorem 7.2 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and suppose that either p is odd or $\mathcal{F}$ is S_3-free. If $\mathcal{F}$ possesses a fixed-point-free automorphism σ of prime order then $\mathcal{F} = \mathcal{F}_P(P)$.

Proof: By Theorem 1.1, if $\mathrm{N}_{\mathcal{F}}(J(P))$ and $\mathrm{C}_{\mathcal{F}}(\mathrm{Z}(P))$ are trivial then $\mathcal{F}$ is trivial. If either of these two subsystems is $\mathcal{F}$ itself then $\mathrm{O}_p(\mathcal{F}) > 1$, so that by induction $\mathcal{F}/\mathrm{O}_p(\mathcal{F})$, which inherits a fixed-point-free automorphism of prime order, is trivial, and $\mathcal{F}$ is soluble; hence by Proposition 7.1, $\mathcal{F} = \mathcal{F}_P(P)$, as needed.

Hence both $\mathrm{N}_{\mathcal{F}}(J(P))$ and $\mathrm{C}_{\mathcal{F}}(\mathrm{Z}(P))$ are proper subsystems of $\mathcal{F}$. As both $J(P)$ and $\mathrm{Z}(P)$ are characteristic subgroup of P, they are σ-invariant, and clearly both subsystems inherit a fixed-point-free action of σ, hence are trivial by induction. Thus Theorem 1.1 implies that $\mathcal{F} = \mathcal{F}_P(P)$, as needed. $\square$

Corollary 7.3 Let G be a finite group admitting a fixed-point-free automorphism σ of prime order. Then G is nilpotent.

Proof: If p is a prime dividing $|G|$, and $P \in \mathrm{Syl}_p(G)$, then σ induces a fixed-point-free automorphism on $\mathcal{F}_P(G)$, whence $\mathcal{F}_P(G) = \mathcal{F}_P(P)$ if p is odd, so that G possesses a normal p-complement H_p for all odd primes p dividing G. If $|G|$ is odd then G is therefore nilpotent, so $|G|$ is even. The intersection of these normal complements H_p is the Sylow 2-subgroup S of G, and clearly G is soluble, since by taking the intersections of successively fewer of the H_p one gets a normal series whose quotients are p-groups for various primes p. Thus $\mathcal{F}_S(G)$ is also soluble, so trivial by Proposition 7.1, and G is nilpotent as its fusion system at all primes are trivial. $\square$

It remains to deal with the case of an insoluble saturated fusion system for the prime 2, which hasn't been disposed of in the above theorems. At the moment this case eludes us: if Q is a σ-invariant subgroup then $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_P(Q)$, but this does not appear to be enough to make much headway.

References

[1] Carles Broto, Natàlia Castellana, Jesper Grodal, Ran Levi, and Bob Oliver, *Subgroup families controlling p-local finite groups*, Proc. London Math. Soc. **91** (2005), 325–354.

[2] Carles Broto, Ran Levi, and Bob Oliver, *The homotopy theory of fusion systems*, J. Amer. Math. Soc. **16** (2003), 779–856.

[3] David A. Craven, *The theory of fusion systems: An algebraic approach*, Cambridge Studies in Advanced Mathematics, vol. 131, Cambridge University Press, Cambridge, 2011.

[4] Antonio Díaz, Adam Glesser, Nadia Mazza, and Sejong Park, *Glauberman's and Thompson's theorems for fusion systems*, Proc. Amer. Math. Soc. **137** (2009), 495–503.

[5] Paul Flavell, *Automorphisms and fusion in finite groups*, J. Algebra **300** (2006), 472.

[6] Radha Kessar and Markus Linckelmann, *On two theorems of Flavell*, Arch. Math. **92** (2009), 1–6.

[7] John Thompson, *Normal p-complements for finite groups*, Math. Z. **72** (1960), 332–354.

[8] ———, *Normal p-complements for finite groups*, J. Algebra **1** (1964), 43–46.