Alperin's Fusion Theorem and Fusion Systems

David A. Craven

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Abstract

This short note provides a new and straightforward proof of the original fusion theorem of Alperin, then considers so-called domestic intersections, which are special types of tame intersections that should play a role in fusion systems, then completes the classification of weak conjugation families, a result essentially due to Puig.

1 Introduction

The theory of fusion systems has grown in an organic way, and so certain subjects have not been given the attention they perhaps deserve.

We start by examining Alperin's fusion theorem for finite groups. The proofs in the literature either follow Alperin's original proof [1] reasonably faithfully (see for example [2] and [4]) or prove strictly weaker results (see for example [7]). Here we give, we believe, the first short and transparent proof of the original version of Alperin's fusion theorem.

We then relate the various versions of Alperin's fusion theorem for fusion systems to the corresponding statements for groups. Broadly speaking, a conjugation family is a collection $\mathcal{Q}$ of subgroups of a fixed Sylow p-subgroup P of G such that, if $g \in G$ maps one subgroup A of P to another, we can write g as a product of elements of normalizers of tame intersections. A weak conjugation family is one where it is not g that can be written this way, but merely the conjugation map c_g of g on A. Weak conjugation families are the correct concept for fusion systems, just as conjugation families are the correct concept for groups.

The corresponding notion for weak conjugation families of tame intersections are *domestic intersections*, intersections $A = P \cap P^{g^{-1}}$, where replacing g by an element that acts the same on A doesn't change the intersection.

Theorem 1.1 Let G be a finite group, and let P be a Sylow p-subgroup of G. The set of domestic intersections lying in P is a weak conjugation family. If H is another finite group with Sylow p-subgroup P such that $\mathcal{F}_P(G) = \mathcal{F}_P(H)$, then the domestic intersections of G lying in P are the domestic intersections of H lying in P; in other words, the set of domestic intersections is an invariant of the fusion system.

In fact, this theorem is proved for saturated fusion systems in general. The set of domestic intersections lies strictly between the set of fully normalized, centric, radical subgroups and the set of fully normalized essential subgroups.

We then classify all weak conjugation families for fusion systems (including those to finite groups) and then conjugation families for finite groups. Both of these are previously known, to Goldschmidt [3] and Puig [6], but here we give a much cleaner proof.

2 Preliminaries

As this paper deals with both groups and fusion systems we will use both terminologies for certain classes of subgroups.

Definition 2.1 Let G be a finite group and let P and Q be Sylow p-subgroups of G.

(i) A subgroup A of P is *extremal* in P with respect to G if $\mathrm{N}_P(A)$ is a Sylow p-subgroup of $\mathrm{N}_P(G)$.

(ii) The intersection $A = P \cap Q$ is *tame* if A is extremal in both P and Q with respect to G.

Since we generally deal with a single fixed Sylow p-subgroup P of G when doing fusion theory, we want a condition on the intersection that only deals with P

Lemma 2.2 Let G be a finite group, and let P be a Sylow p-subgroup of G. Let $g \in G$, and write $A = P \cap P^{g^{-1}}$. The intersection A is tame if and only if both A and A^g are extremal in P with respect to G.

Proof: If A is a subgroup of P, then $\mathrm{N}_P(A^g)$ is a Sylow p-subgroup of $\mathrm{N}_G(A^g)$ if and only if $\mathrm{N}_{P^{g^{-1}}}(A)$ is a Sylow p-subgroup of $\mathrm{N}_G(A)$; i.e., A^g is extremal in P with respect to G if and only if A is extremal in $P^{g^{-1}}$ with respect to G. The result now follows. $\square$

Traditionally, the definition of an extremal subgroup was that Q is extremal if $|\,\mathrm{N}_P(Q)| \geqslant |\,\mathrm{N}_P(R)|$ whenever Q and R are conjugate in G. This is equivalent to the definition given here by the following lemma.

Lemma 2.3 Let G be a finite group, and let P be a Sylow p-subgroup of G. Let A be a subgroup of P.

(i) There exists an extremal subgroup of P that is G-conjugate to A.

(ii) If C is an extremal subgroup of P that is G-conjugate to A, there exists an element $g \in G$ such that $A^g = C$ and $\mathrm{N}_P(A)^g \leqslant \mathrm{N}_P(C)$. Furthermore, if h is any other element of G such that $A^h = C$, then g may be chosen so that $h^{-1}g \in \mathrm{O}^{p'}(\mathrm{N}_G(C))$.

Proof: Let Q be a Sylow p-subgroup of G containing $\mathrm{N}_P(A)$ and such that $\mathrm{N}_Q(A)$ is a Sylow p-subgroup of $\mathrm{N}_G(A)$. Let g be such that $Q^g = P$. Notice that $C = A^g \leqslant P$, and $\mathrm{N}_P(C) = (\mathrm{N}_Q(A))^g$ is a Sylow p-subgroup of $\mathrm{N}_G(C)$, so that C is extremal. This proves (i).

Let C be any extremal subgroup of P that is G-conjugate to A, and let Q be a Sylow p-subgroup of G as above. Let h be any element of G such that $A^h = C$; clearly $\mathrm{N}_G(A)^h = \mathrm{N}_G(C)$ and so, since $\mathrm{N}_P(C)$ is a Sylow p-subgroup of $\mathrm{N}_G(C)$, there exists $x \in \mathrm{N}_G(C)$ such that $\left(\mathrm{N}_P(A)^h\right)^x \leqslant \mathrm{N}_P(C)$. Let $g = hx$. Furthermore, since all Sylow p-subgroups of G are conjugate in $\mathrm{O}^{p'}(G)$, we may choose x to lie in $\mathrm{O}^{p'}(\mathrm{N}_G(C))$. We therefore have that $\mathrm{N}_P(A)^g \leqslant \mathrm{N}_P(C)$ and $h^{-1}g = x \in \mathrm{O}^{p'}(\mathrm{N}_G(C))$, as claimed. $\square$

Definition 2.4 Let G be a finite group and let P be a Sylow p-subgroup of G.

(i) A set of subgroups of P is called *family*. Let $\mathcal{Q}$ denote a family.

(ii) The set $\mathcal{Q}$ is a *conjugation family* if, whenever A and B are subgroups of P and $g \in G$ is such that $A^g = B$, there exist (for $1 \leqslant i \leqslant n$) $Q_i \in \mathcal{Q}$ and $x_i \in \mathrm{N}_G(Q_i)$, such that $g = x_1 \dots x_n$ and $A^{x_1 \dots x_i} \leqslant Q_{i+1}$ for all $0 \leqslant i \leqslant n-1$.

(iii) The set $\mathcal{Q}$ is a *weak conjugation family* if, whenever A and B are subgroups of P and $g \in G$ is such that $A^g = B$, there exist (for $1 \leqslant i \leqslant n$) $Q_i \in \mathcal{Q}$ and $x_i \in \mathrm{N}_G(Q_i)$ such that conjugation by g and conjugation by $x_1 \dots x_n$ agree on A, and $A^{x_1 \dots x_i} \leqslant Q_{i+1}$ for all $0 \leqslant i \leqslant n-1$.

By the nature of fusion systems, one may only prove that collections of subgroups are weak conjugation families, and so no proof of the original Alperin's fusion theorem (which involves conjugation families) can be obtained using fusion systems.

Definition 2.5 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. A collection $\mathcal{Q}$ of subgroups of P is a *weak conjugation family* if $\mathcal{F} = \langle \mathrm{Aut}_{\mathcal{F}}(Q) : Q \in \mathcal{Q} \rangle$.

Let $\mathcal{Q}$ be a family, let A and B be subsets of P and $g \in G$ be such that $A^g = B$. We write $A \xrightarrow{g} B$ with respect to $\mathcal{Q}$ if there exist, for $1 \leqslant i \leqslant n$, $Q_i \in \mathcal{Q}$, and elements $x_i \in \mathrm{N}_G(Q_i)$, such that

(i) $g = x_1 x_2 \dots x_n$;

(ii) $A^{x_1 x_2 \dots x_i} \subseteq Q_{i+1}$ for all $0 \leqslant i \leqslant n-1$.

If the family under consideration is clear from the context, we will drop 'with respect to $\mathcal{Q}$'. Notice that $\mathcal{Q}$ is a conjugation family if and only if $A \xrightarrow{g} B$ for all such A, B and g. Note that for $A, B, C \subseteq P$, if $A \xrightarrow{g} B$ and $B \xrightarrow{h} C$ then $A \xrightarrow{gh} C$ and $C \xrightarrow{h^{-1}} B$, and if $A \xrightarrow{g} B$ and $C \subseteq A$ then $C \xrightarrow{g} C^g$.

We end by showing that if $\mathcal{Q}$ is a conjugation family then the elements x_i may be chosen to be p-elements, when the Q_i are proper subgroups of P.

Proposition 2.6 Let G be a finite group and let P be a Sylow p-subgroup of G. Let $\mathcal{Q}$ be a conjugation family. If $A, B \subseteq P$, and $g \in G$ are such that $A \xrightarrow{g} B$, then we may choose the x_i to be p-elements of $\mathrm{N}_G(Q_i)$ whenever $Q_i < P$.

Proof: Of course, it suffices to prove the result in the case where $A = B \in \mathcal{Q}$ and so $g \in \mathrm{N}_G(A)$. We may assume by conjugating that A is extremal in P with respect to G. Let $X = \mathrm{N}_G(A)$ and $Y = \mathrm{O}^{p'}(\mathrm{N}_G(A))$. As $\mathrm{N}_P(A)$ is a Sylow p-subgroup of $\mathrm{O}^{p'}(A)$, by a Frattini argument $X = Y\,\mathrm{N}_X(\mathrm{N}_P(A))$. Clearly every element of Y may be written as a product of p-elements (as $\mathrm{O}^{p'}(X)$ is generated by p-elements), and an element h of $\mathrm{N}_X(\mathrm{N}_P(A))$ that normalizes $\mathrm{N}_P(A)$, hence by induction on $|P : A|$ is expressible as a product of p-elements and elements that normalize P; the result follows. $\square$

3 A Proof of Alperin's Fusion Theorem

Here we will give a short proof of Alperin's fusion theorem.

Theorem 3.1 (Alperin's fusion theorem [1]) Let G be a finite group, and let P be a Sylow p-subgroup of G. Let A and B be two subsets of P such that $A = B^g$. There exist Sylow p-subgroups $S_1, \dots, S_n$ of G and elements $x_1, \dots, x_n$ of G such that

(i) $g = x_1 x_2 \dots x_n$;

(ii) $P \cap S_i$ is a tame intersection for all i;

(iii) x_i is a p-element of $\mathrm{N}_G(P \cap S_i)$ unless $P = S_i$, for all i;

(iv) $A^{x_1 x_2 \dots x_i}$ is a subset of $P \cap S_{i+1}$ for all $0 \leqslant i \leqslant n-1$.

Proof: Let $\mathcal{Q}$ denote the set of tame intersections, which includes P. Let A and B be subsets of P such that $A^g = B$ for some $g \in G$, and we must show that $A \xrightarrow{g} B$ (with respect to $\mathcal{Q}$).

We may assume that A (and hence B) is the intersection of two Sylow p-subgroups of G, since if $P^{g^{-1}} \cap P \xrightarrow{g} P \cap P^g$ then since $A \subseteq P^{g^{-1}} \cap P$ we have that $A \xrightarrow{g} B$. We proceed by induction on $m = |P : A|$. If $m = 1$ then $A = P$, whence $g \in \mathrm{N}_G(P)$, and $P \xrightarrow{g} P$, so we may suppose that $m > 1$.

By hypothesis $A < P$, and so $A < \mathrm{N}_P(A)$. Choose a subgroup C of P that is G-conjugate to A and extremal in P with respect to G. By Lemma 2.3, we may choose $h \in G$ such that $A^h = C$ and $\mathrm{N}_P(A)^h \leqslant \mathrm{N}_P(C)$, so that by induction $\mathrm{N}_P(A) \xrightarrow{h} (\mathrm{N}_P(A))^h$ and hence $A \xrightarrow{h} C$. Similarly, there exists $k \in G$ such that $B \xrightarrow{k} C$. We see therefore that $x = h^{-1}gk$ normalizes C. If $X = P \cap P^{x^{-1}}$ properly contains C then by induction $X \xrightarrow{x} X^x$ so $C \xrightarrow{x} C$, whence

$$A \xrightarrow{h} C \xrightarrow{x} C \xrightarrow{k^{-1}} B,$$

and so $A \xrightarrow{g} B$. Therefore $X = C$, so that C is the intersection of P and $P^{x^{-1}}$. As C and $C = C^x$ are both extremal in P with respect to G, by Lemma 2.2 we see that $C = P \cap P^{x^{-1}}$ is a tame intersection, so in $\mathcal{Q}$. Hence $C \xrightarrow{x} C$ and the proof is complete. $\square$

4 Domestic Intersections

Tame intersections are the correct object to consider for conjugation families, but for weak conjugation families we do not need so many of them. If $A = P \cap P^{g^{-1}}$ is a tame intersection then there is an induced map $c_g : A \to A^g$ inside P. Furthermore, A is the largest subgroup of P for which g induces a map by conjugation. However, by replacing g with a different element inducing the same map as c_g on A (i.e., xg for $x \in \mathrm{C}_G(A)$) we might have that $P \cap P^{(xg)^{-1}}$ strictly contains A.

Definition 4.1 Let G be a finite group, let P be a Sylow p-subgroup of G, and let $g \in G$. The intersection $A = P \cap P^{g^{-1}}$ is *domestic* if it is tame, and whenever $x \in \mathrm{C}_G(A)$ we have $P \cap P^{(xg)^{-1}} = A$.

Domestic intersections can be detected in the fusion system, unlike tame intersections.

Proposition 4.2 Let G be a finite group, and let P be a Sylow p-subgroup of G. Write $\mathcal{F} = \mathcal{F}_P(G)$. A subgroup A of P is a domestic intersection if and only if A is fully $\mathcal{F}$-normalized and there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(A, P)$ such that $N_\phi = A$ and $A\phi$ is also fully $\mathcal{F}$-normalized.

Proof: Suppose that $A = P \cap P^{g^{-1}}$ is a domestic intersection. If $\psi : X \to Y$ is any morphism in $\mathcal{F} = \mathcal{F}_P(G)$ such that $A \leqslant X$ and $\psi|_A = \phi$ then $\psi = c_h$ for some $h = xg$ with $x \in \mathrm{C}_G(A)$. By assumption, $P \cap P^{h^{-1}} = A$, so $X = A$, as needed, proving one direction.

Conversely, suppose that there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(A, P)$ such that $N_\phi = A$ and $A\phi$ is also fully $\mathcal{F}$-normalized. By Lemma 2.2, A is a tame intersection. If $h = xg$ for some $x \in \mathrm{C}_G(A)$ then $c_g = c_h$, so $\mathrm{N}_P(A) \cap \mathrm{N}_P(A)^{h^{-1}} = A$; hence $P \cap P^{h^{-1}} = A$, proving that A is a domestic intersection. $\square$

The statement that there exists $\phi \in \mathrm{Hom}_{\mathcal{F}}(A, P)$ such that $N_\phi = A$ means that firstly, A is $\mathcal{F}$-centric, and secondly that $\mathrm{Out}_P(A) \cap \mathrm{Out}_P(A)^{\phi^{-1}} = 1$, so in particular A is $\mathcal{F}$-radical. In fact, we get the following result.

Proposition 4.3 Let A be a subgroup of P.

(i) If A is a domestic intersection then A is fully $\mathcal{F}$-normalized, $\mathcal{F}$-centric, and $\mathcal{F}$-radical.

(ii) If A is fully $\mathcal{F}$-normalized and $\mathcal{F}$-essential then A is a domestic intersection.

Proof: We have already proved (i) above, so we prove (ii). Suppose that A is $\mathcal{F}$-essential and fully $\mathcal{F}$-normalized. Since A is $\mathcal{F}$-essential, there exists a strongly p-embedded subgroup M of $\mathrm{Out}_{\mathcal{F}}(A)$ containing $\mathrm{Out}_P(A)$. Let X be any Sylow p-subgroup of $\mathrm{Out}_{\mathcal{F}}(A)$ not contained in M, and notice that $X \cap \mathrm{Out}_P(A) = 1$. Let $\psi \in \mathrm{Aut}_{\mathcal{F}}(A)$ be such that $\mathrm{Aut}_P(A)^\psi / \mathrm{Inn}(A) = X$. Clearly A and $A\psi = A$ are fully $\mathcal{F}$-normalized, $N_\psi = A$. Proposition 4.2 now proves that A is a domestic intersection. $\square$

An interesting question is whether there are fully normalized, centric, radical subgroups of a Sylow p-subgroup of G that are not domestic intersections. This is equivalent to asking whether, in a finite group G with Sylow p-subgroup P, there exists $g \in G$ such that $P \cap P^g = \mathrm{O}_p(G)$. In [5], it was proved that $\mathrm{O}_p(G)$ is the intersection of *three* Sylow p-subgroups of G, and a fairly restrictive set of conditions on when it is not the intersection of two are obtained: in particular, p is one of 2, 3, or a Mersenne prime. The easiest example of such a situation is $G = S_8$ and $p = 2$. Hence the set of domestic intersections lies strictly between the set of fully normalized, centric, radical subgroups and the set of fully normalized essential subgroups.

5 Conjugation Families and Essential Subgroups

Theorem 5.1 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{Q}$ be a collection of subgroups of P. Then $\mathcal{Q}$ is a weak conjugation family for $\mathcal{F}$ if and only if $\mathcal{Q}$

contains a representative from each $\mathcal{F}$-conjugacy class of essential subgroups of P, together with P if $\mathrm{Aut}_{\mathcal{F}}(P)$ is not a p-group.

The proof of this theorem will involve a series of lemmas. Since this is all well known for fusion systems, we do not prove the result in great detail.

Lemma 5.2 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let $\mathcal{Q}$ be a weak conjugation family for $\mathcal{F}$. If $\mathcal{Q}'$ is a subset of $\mathcal{Q}$ such that every $\mathcal{F}$-conjugacy class of subgroups of P represented in $\mathcal{Q}$ is represented in $\mathcal{Q}'$, then $\mathcal{Q}'$ is a weak conjugation family.

Proof: This follows easily from the well-known fact that the $\mathcal{F}$-conjugacy classes containing Q is the same as the $\mathcal{E}$-conjugacy class, where $\mathcal{E}$ is generated by $\mathrm{Aut}_{\mathcal{F}}(R)$ for $|R| > |Q|$. $\square$

The next lemma follows almost immediately from the definition of a strongly p-embedded subgroup, and its proof is omitted.

Lemma 5.3 Let G be a finite group, let P be a Sylow p-subgroup of G, and let Q be a normal p-subgroup of G. Let X denote the subgroup generated by all $g \in G$ such that there exists $Q < R \leqslant P$ such that $R^g \leqslant P$. The group G/Q has a strongly p-embedded subgroup if and only if $X < G$.

This lemma does, however, imply an important result.

Proposition 5.4 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a subgroup of P. The collection $\mathcal{Q}$ of all subgroups of P not $\mathcal{F}$-conjugate to Q is a weak conjugation family if and only if Q is neither essential, nor P if $\mathrm{Aut}_{\mathcal{F}}(P)$ is not a p-group.

Proof: If $Q = P$ then the result holds, so we may assume that $Q < P$. We may also assume that Q is fully normalized. Let $\mathcal{E}$ be the subsystem of $\mathcal{F}$ generated by $\mathrm{Aut}_{\mathcal{F}}(R)$ for all $R \in \mathcal{Q}$. Clearly, $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{E}}(Q)$ if and only if $\mathcal{Q}$ is a weak conjugation family. As However, if $\mathrm{Aut}_R(Q)^\phi \leqslant \mathrm{Aut}_P(Q)$ for some $Q < R \leqslant \mathrm{N}_P(Q)$ and $\phi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, then as $R > Q$ and Q is fully normalized, ϕ extends to $\psi \in \mathrm{Aut}_{\mathcal{F}}(R)$, and $\psi \in \mathrm{Aut}_{\mathcal{E}}(R)$ as $R \in \mathcal{Q}$; this implies that, if X denotes the subgroup generated by all such $\phi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, that $X = \mathrm{Aut}_{\mathcal{E}}(Q)$.

If Q is not essential, $X = \mathrm{Aut}_{\mathcal{E}}(Q)$ by Lemma 5.3 and $\mathcal{E} = \mathcal{F}$, completing the proof of one direction. Hence we may assume that Q is essential, and let $\phi \in \mathrm{Aut}_{\mathcal{F}}(Q) \setminus X$. For a contradiction, assume that $\mathrm{Aut}_{\mathcal{E}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)$. Hence there exist subgroups $R_i > Q$ of P and elements $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(R_i)$ such that $\phi = \psi_1 \dots \psi_n$. Write $Q_0 = Q$ and $Q_i = Q_{i-1}^{\psi_i}$, so that $Q_n = Q$. Since Q is fully normalized, there exists $\chi_i \in \mathrm{Hom}_{\mathcal{F}}(Q_i, Q)$ such that $\mathrm{Aut}_P(Q_i)^{\chi_i} \leqslant \mathrm{Aut}_P(Q)$. Set $\theta_0 = \theta_n = 1$, and $\theta_i = \chi_{i-1}^{-1}\psi_i\chi_i$, so that $\theta_i \in \mathrm{Aut}_{\mathcal{F}}(Q)$; notice that $\theta_1 \dots \theta_n = \phi$. As $R_i > Q_i$, $\mathrm{Aut}_{R_i}(Q_{i-1}) > \mathrm{Inn}(Q_i)$, and $\mathrm{Aut}_{R_i}(Q_{i-1})^{\psi_i} = \mathrm{Aut}_{R_i}(Q_i)$;

this implies that θ_i maps some proper overgroup of Q in $\mathrm{N}_P(Q)$ to some other overgroup of Q in $\mathrm{N}_P(Q)$, so that $\theta_i \in X$ for each i (with $\phi = \theta_1 \dots \theta_n$), a contradiction to the fact that $\phi \in \mathrm{Aut}_{\mathcal{F}}(Q) \setminus X$. This completes the proof. $\square$

Proposition 5.5 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two collections of subgroups of P, closed under $\mathcal{F}$-conjugation. If $\mathcal{Q}$ and $\mathcal{Q}'$ are weak conjugation families then $\mathcal{Q} \cap \mathcal{Q}'$ is a weak conjugation family.

Proof: Write $\mathcal{R} = \mathcal{Q} \cap \mathcal{Q}'$, and suppose that $\mathcal{R}$ is not a weak conjugation family. Write $\mathcal{D}$ for the subsystem of $\mathcal{F}$ generated by $\mathrm{Aut}_{\mathcal{F}}(Q)$ for $Q \in \mathcal{R}$. Choose $A \leqslant P$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(A, P)$, and we will show that $\phi \in \mathcal{D}$. Proceed by induction on $|P : A|$: if A does not lie in $\mathcal{R}$, then it does not lie in (without loss of generality) $\mathcal{Q}$. Hence, since $\mathcal{Q}$ is a weak conjugation family, there is a sequence of proper overgroups R_i of $\mathcal{F}$-conjugates of Q and elements $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(R_i)$ such that $\psi_1 \dots \psi_n = \phi$. As $|R_i| > |Q|$, by induction each ψ_i lies in $\mathcal{D}$, whence ϕ lies in $\mathcal{D}$. This completes the proof. $\square$

The proof of Theorem 5.1 is now clear, since by Propositions 5.4 and 5.5 we see that the union of all essential subgroups is a conjugation family, and Lemma 5.2 completes the proof.

6 Conjugation Families and Quasi-Essential Subgroups

Just as every weak conjugation family must contain a member from every $\mathcal{F}$-conjugacy class of essential subgroups, we get a similar result for finite groups, and G-conjugacy classes of *quasi-essential* subgroups.

Definition 6.1 Let G be a finite group and let P be a Sylow p-subgroup of G. A subgroup Q of P is *quasi-essential* if $\mathrm{N}_G(Q)/Q$ contains a strongly p-embedded subgroup.

Since a finite group X has a strongly p-embedded subgroup if and only if $X/\mathrm{O}_{p'}(X)$ does, a subgroup Q is $\mathcal{F}_P(G)$-essential if and only if it is p-centric and quasi-essential. The quasi-essential subgroups of P play the same role for groups as the essential subgroups for fusion systems.

Theorem 6.2 Let G be a finite group and let P be a Sylow p-subgroup of G. A collection $\mathcal{Q}$ of subgroups of P is a conjugation family if and only if it contains a representative from every G-conjugacy class of quasi-essential subgroups, together with P if $\mathrm{N}_G(P) > P\,\mathrm{C}_G(P)$.

The proof of this theorem follows the same strategy as the corresponding theorem for fusion systems.

Lemma 6.3 Let G be a finite group, and let P be a Sylow p-subgroup of G. Let $\mathcal{Q}$ be a conjugation family. If $\mathcal{Q}'$ is a subset of $\mathcal{Q}$ such that every G-conjugacy class of subgroups of P represented in $\mathcal{Q}$ is represented in $\mathcal{Q}'$, then $\mathcal{Q}'$ is a conjugation family.

Proof: We need only prove that, if $Q \in \mathcal{Q} \setminus \mathcal{Q}'$, then $Q \xrightarrow{g} Q$ with respect to $\mathcal{Q}'$ for all $g \in \mathrm{N}_G(Q)$. To see this, let $R \in \mathcal{Q}'$ be a subgroup of P that is G-conjugate to Q, which exists by assumption. There exists $x \in G$ such that $Q^x = R$ and hence $Q \xrightarrow{x} R$ with respect to $\mathcal{Q}$. By replacing x with another element that conjugates Q to R, we may assume that x is a product of elements x_i that normalize subgroups of P of smaller index than Q, whence by induction $Q \xrightarrow{x} R$ with respect to $\mathcal{Q}'$. Finally,

$$Q \xrightarrow{x} R \xrightarrow{g^x} R \xrightarrow{x^{-1}} Q$$

with respect to $\mathcal{Q}$, so that $Q \xrightarrow{g} Q$ with respect to $\mathcal{Q}$, as claimed. □

Just as in the previous section, Lemma 5.3 provides the key step.

Proposition 6.4 Let G be a finite group and let P be a Sylow p-subgroup of G. Let Q be a subgroup of P. The collection $\mathcal{Q}$ of all subgroups of P not G-conjugate to Q is a conjugation family if and only if Q is not quasi-essential, or P if $\mathrm{N}_G(P) > P\,\mathrm{C}_G(P)$.

Proof: If $Q = P$ then the result holds, so we may assume that $Q < P$. We may also assume that Q is extremal with respect to G. If $Q \xrightarrow{g} Q$ with respect to $\mathcal{Q}$ for all $g \in \mathrm{N}_G(Q)$ then clearly $\mathcal{Q}$ is a conjugation family, and so it suffices to show this case. However, if $R^x \leqslant \mathrm{N}_P(Q)$ for some $Q < R \leqslant \mathrm{N}_P(Q)$ and $x \in \mathrm{N}_G(Q)$, then as $R > Q$ we must have that $R \xrightarrow{x} R^g$; this implies that, if X denotes the subgroup generated by all such x, that $Q \xrightarrow{g} Q$ for all $g \in X$.

If Q is not quasi-essential, $X = G$ by Lemma 5.3, completing the proof of one direction. Hence we may assume that Q is quasi-essential, and let $g \in G \setminus X$. For a contradiction, assume that $Q \xrightarrow{g} Q$. Hence there exist subgroups $R_i > Q$ of P and elements $x_i \in \mathrm{N}_G(R_i)$ such that $g = x_1 \dots x_n$. Write $Q_0 = Q$ and $Q_i = Q_{i-1}^{x_i}$, so that $Q_n = Q$. Since Q is extremal in P, there exists $h_i \in G$ such that $\mathrm{N}_P(Q_i)^{h_i} \leqslant \mathrm{N}_P(Q)$ and $Q_i^{h_i} = Q$. Set $y_0 = y_n = 1$, and $y_i = h_{i-1}^{-1} x_i h_i$, so that $y_i \in \mathrm{N}_G(Q)$; notice that $y_1 \dots y_n = g$. As $R_i > Q_i$, $\mathrm{N}_{R_i}(Q_{i-1}) > Q_i$, and $\mathrm{N}_{R_i}(Q_{i-1})^{x_i} = \mathrm{N}_{R_i}(Q_i)$; this implies that y_i conjugates some proper overgroup of Q in $\mathrm{N}_P(Q)$ to some other overgroup of Q in $\mathrm{N}_P(Q)$, so that $y_i \in X$ for each i (with $g = y_1 \dots y_n$), a contradiction to the fact that $g \in G \setminus X$. This completes the proof. □

Proposition 6.5 Let G be a finite group and let P be a Sylow p-subgroup of G. Let $\mathcal{Q}$ and $\mathcal{Q}'$ be two collections of subgroups of P, closed under G-conjugation. If $\mathcal{Q}$ and $\mathcal{Q}'$ are conjugation families then $\mathcal{Q} \cap \mathcal{Q}'$ is a conjugation family.

Proof: Write $\mathcal{R} = \mathcal{Q} \cap \mathcal{Q}'$, and suppose that $\mathcal{R}$ is not a conjugation family. Choose $A \leqslant P$ and $g \in G$ such that $A^g \leqslant P$, and we will show that $A \xrightarrow{g} B$ with respect to $\mathcal{R}$. Proceed by induction on $|P : A|$; if A does not lie in $\mathcal{R}$, then it does not lie in (without loss of generality) $\mathcal{Q}$. Hence, since $\mathcal{Q}$ is a conjugation family there is a sequence of proper overgroups R_i of G-conjugates of Q and elements $x_i \in \mathrm{N}_G(R_i)$ such that $x_1 \dots x_n = g$. As $|R_i| > |Q|$, by induction $R_i \xrightarrow{x_i} R_i$ with respect to $\mathcal{R}$, whence $A \xrightarrow{g} B$ with respect to $\mathcal{R}$. This completes the proof. $\square$

The proof of Theorem 6.2 is now clear, since by Propositions 6.4 and 6.5 we see that the union of all quasi-essential classes is a conjugation family, and Lemma 6.3 completes the proof.

This yields the following corollary.

Corollary 6.6 Let G be a finite group and let P be a Sylow p-subgroup of G. Every quasi-essential class contains a tame intersection.

Proof: The collection of all tame intersections forms a conjugation family, as does the collection of all quasi-essential subgroups, so the set of quasi-essential tame intersections also form a conjugation family. Theorem 6.2 completes the proof. $\square$

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