Fusion Systems: Group theory, representation theory, and topology

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Michaelmas Term, 2008

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Preface

It is difficult to pinpoint the origins of the theory of fusion systems: it could be argued that they stretch back to Burnside and Frobenius, with arguments about the fusion of p-elements of finite groups. Another viewpoint is that it really started with the theorems on fusion in finite groups, such as Alperin's fusion theorem, or Grün's theorems.

We will take as the starting point the important paper of Solomon [23], which proves that, for a Sylow 2-subgroup P of $\mathrm{Spin}_7(3)$, there is a particular pattern of the fusion of p-elements in P that, while not internally inconsistent, is not consistent with living inside a finite group. This is the first instance where the fusion of p-elements looks fine on its own, but is incompatible with coming from a finite group.

Unpublished work of Puig during the 1990s (some of which is collected in [21]), together with work of Alperin–Broué [2], is the basis for constructing a fusion system for a p-block of a finite group. It was with Puig's work where the axiomatic foundations of fusion systems started, and where some of the fundamental notions begin.

Along with the representation theory, topology played an important role in the development of the theory: Benson [7] constructed a topological space that should be the p-completed classifying space of a finite group whose fusion pattern matched that which Solomon considered. Since such a group does not exist, this space can be thought of as the shadow cast by an invisible group. Benson predicted that this topological space is but one facet of a general theory, a prediction that was confirmed with the development of p-local finite groups.

Although we will define a p-local finite group properly in Chapter 6, it can be thought of as some data describing a p-completed classifying space of a fusion system. In the case where the fusion system arises from a finite group, the corresponding p-local finite group decribes the normal p-completed classifying space.

In this direction, we have Oliver's proof [19] [20] of the Martino–Priddy conjecture [18], which states that two finite groups have homotopy equivalent p-completed classifying spaces if and only if the fusion systems are isomorphic. The topological considerations have fuelled development in the algebraic aspects of fusion systems and vice versa, and the two viewpoints are intertwined.

As this is a young subject, still in development, the foundations of the theory have not yet been solidified; indeed, there is some debate as to the correct *definition* of a fusion system! The definition of a normal subsystem is also under discussion, and which definition is used often indicates the intended applications of the theory. Since group theorists, representation theorists, and topologists all converge on this area, there are several different conventions and styles, as well as approaches.

The choice of definitions and conventions has been influenced by the background of the author: as a group representation theorist, the conventions here will be the standard algebra conventions, rather than topology conventions. In particular, homomorphisms will be composed from left to right.

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March 28, 2009

Chapter 1

Fusion in Finite Groups

The fusion of elements of prime power order in a finite group is the source of many deep theorems in finite group theory. In this chapter we will briefly survey this area, and use this theory to introduce the notion of a fusion system on a finite group.

1.1 Control of Fusion

We begin with a famous theorem.

Theorem 1.1 (Burnside) Let G be a finite group with abelian Sylow p-subgroups. Let x and y be two elements in a Sylow p-subgroup P. If x and y are G-conjugate then they are $\mathrm{N}_G(P)$-conjugate.

Proof: Suppose that x and y are elements of $P \in \mathrm{Syl}_p(G)$ that are conjugate in G, so that $x = y^g$. Thus

$$P^g \leqslant \mathrm{C}_G(x)^g = \mathrm{C}_G(x^g) = \mathrm{C}_G(y).$$

Thus both P and P^g are Sylow p-subgroups of $\mathrm{C}_G(y)$. Therefore there is some $h \in \mathrm{C}_G(y)$ such that $P^{gh} = P$, and so $gh \in \mathrm{N}_G(P)$. Moreover, $x^{gh} = y^h = y$, as required. $\square$

This theorem is a statement about the *fusion* of P-conjugacy classes in G.

Definition 1.2 Let G be a finite group, and let H and K be subgroups of G with $H \leqslant K$.

(i) Let g and h be elements of H that are not conjugate in H. Then g and h are *fused* in K if g and h are conjugate by an element of K. Similarly, we say that two subgroups or two conjugacy classes are fused if they satisfy the obvious condition.

(ii) The subgroup K is said to *control fusion in H with respect to G* if, whenever g and h are fused in G, they are fused in K. (This is equivalent to the fusion of conjugacy classes.)

(iii) The subgroup K is said to *control G-fusion in H* if, whenever two subgroups A and B are conjugate via a conjugation map $\theta_g : A \to B$ for some $g \in G$, then there is some $k \in K$ such that θ_g and θ_k agree on A. (This is stronger than simply requiring any two subgroups conjugate in G to be conjugate in K.)

Given these definitions, Theorem 1.1 has the following restatement.

Corollary 1.3 Let G be a finite group with abelian Sylow p-subgroups, and let P be such a Sylow p-subgroup. Then $\mathrm{N}_G(P)$ controls fusion in P with respect to G.

In fact, we have the following.

Proposition 1.4 Let G be a finite group with an abelian Sylow p-subgroup P. Then $\mathrm{N}_G(P)$ controls G-fusion in P.

Proof: Let A and B be subgroups of P, and suppose that there is some $g \in G$ such that $A^g = B$, via $\theta_g : A \to B$. Thus

$$P^g \leqslant \mathrm{C}_G(A)^g = \mathrm{C}_G(A^g) = \mathrm{C}_G(B).$$

Thus both P and P^g are Sylow p-subgroups of $\mathrm{C}_G(B)$. Thus there is a $h \in \mathrm{C}_G(B)$ such that $P^{gh} = P$. Then $gh \in \mathrm{N}_G(P)$ and we have

$$x^{gh} = (x^g)^h = x^g,$$

and so $\theta_{gh} = \theta_g$ on P, as required. □

What we are saying is that any fusion inside a Sylow p-subgroup P of a finite group must take place inside its normalizer, at least if P is abelian. In general, this is not true.

Example 1.5 Let G be the group $\mathrm{GL}_3(2)$, the simple group of order 168. This group has a dihedral Sylow 2-subgroup P, generated by the two matrices

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Note that x and y are both *involutions*; i.e., have order 2. In $\mathrm{GL}_3(2)$, all of the twenty-one involutions are conjugate, but this cannot be true in $\mathrm{N}_G(P)$, since we claim that $\mathrm{N}_G(P) = P$.

To see this, notice that $\operatorname{Aut}(P)$ has order 8, and $\mathrm{N}_G(P)/\mathrm{C}_G(P)$ is a subgroup of $\operatorname{Aut}(P)$. Thus $\mathrm{N}_G(P)/\mathrm{C}_G(P)$, and hence $\mathrm{N}_G(P)$, is a 2-group.

In fact, there is no normalizer of a p-subgroup – except the normalizer of the identity – in which x and y are conjugate. However, all is not lost; the centre of P has order 2, and is generated by the element

$$z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $Q_1 = \langle x, z\rangle$, and let $N_1 = \mathrm{N}_G(Q_1)$. Then N_1 is isomorphic with the symmetric group on 4 letters, and so inside here the normal subgroup – the Klein four group – has the property that all of its non-identity elements are conjugate in the overgroup. Therefore x and z are conjugate in N_1.

Similarly, write $Q_2 = \langle y, z\rangle$ and $N_2 = \mathrm{N}_G(Q_2)$. Then the same statements apply, and so y and z are conjugate inside N_2. Thus x and y are conjugate, via z, inside normalizers of non-identity 2-subgroups of a particular Sylow 2-subgroup.

This idea of fusion of p-elements not being controlled by a single subgroup, but two elements being conjugate ‘in stages’ by a collection of subgroups is important, and is the basis of Alperin’s fusion theorem, which we will see in Section 3.4 later in this chapter.

The notion of fusion, and control of fusion, is what is interesting for us, and we will explore the fusion and control of fusion in Sylow p-subgroups of finite groups, and more abstractly with the notion of fusion systems. For a group, we give the definition of a fusion system now.

Definition 1.6 Let G be a finite group and let P be a Sylow p-subgroup of G. Then the *fusion system* of G on P is the category $\mathcal{F}_P(G)$, whose objects are all subgroups of P and whose morphism set is

$$\operatorname{Hom}_{\mathcal{F}_P(G)}(A, B) = \operatorname{Hom}_G(A, B),$$

the set of all (not necessarily surjective) maps $A \to B$ induced by conjugation by elements of G. The composition of morphisms is composition of maps.

This definition is meant to capture the notion of fusion of p-elements in the group G. We will see such a fusion system in an example.

Example 1.7 Let G be the group $\mathrm{GL}_3(2)$, considered in Example 1.5, and let P be the Sylow 2-subgroup given there, with the elements x, y and z as constructed. Then P is isomorphic with D_8, so $\mathcal{F}_P(P)$ is simply all of the conjugation actions given by elements of

P. For example, we have the (not surjective) map $\phi : \langle x \rangle \mapsto \langle x, z \rangle$ sending x to xz; this is realized by conjugation by y.

Consider the fusion system $\mathcal{F}_P(G)$, which contains $\mathcal{F}_P(P)$. We will simply describe the bijective maps, since all injective maps in $\mathrm{Hom}_G(A, B)$ are bijections followed by inclusions. There are bijections $\langle g \rangle \mapsto \langle h \rangle$, where g and h are involutions. The two elements of order 4 are conjugate in P, so there is a map $\langle xy \rangle \to \langle xy \rangle$ sending xy to $(xy)^3$. Finally, there are maps of the V_4 subgroups, which we need to consider now. Let $Q_1 = \langle x, z \rangle$ and $Q_2 = \langle y, z \rangle$.

We first consider the maps in $\mathrm{Hom}_{\mathcal{F}_P(G)}(Q_1, Q_1) = \mathrm{Aut}_{\mathcal{F}_P(G)}(Q_1)$. Since $\mathrm{N}_G(Q_1)$ is the symmetric group, and $\mathrm{C}_G(Q_1) = Q_1$, we must have that $\mathrm{Aut}_{\mathcal{F}_G(P)}(Q_1) = \mathrm{Aut}_G(Q_1)$ is isomorphic with S_3, and so is the full automorphism group. (Similarly, $\mathrm{Aut}_{\mathcal{F}_P(G)}(Q_2) = \mathrm{Aut}(Q_2)$.) If ϕ is any map $Q_1 \to Q_2$ in $\mathcal{F}_P(G)$, then by composing with a suitably chosen automorphism of Q_2, we get all possible isomorphisms $Q_1 \to Q_2$. This would include the map ϕ where $\phi : x \mapsto y$ and $\phi : z \mapsto z$. Then x and y would be conjugate in $\mathrm{C}_G(z) = \mathrm{N}_G(P) = P$, and this is not true. Therefore there are no maps between Q_1 and Q_2.

This shows that, although all of the non-identity elements in Q_1 are conjugate to all non-identity elements in Q_2 in $\mathcal{F}_P(G)$, the subgroups Q_1 and Q_2 are not isomorphic in $\mathcal{F}_P(G)$. This is why we take all *subgroups* of P in the fusion system, rather than merely all elements.

The fusion system is meant to capture the concept of control of fusion, and indeed it does.

Proposition 1.8 Let G be a finite group and let P be a Sylow p-subgroup. Let H be a subgroup of G containing P. Then H controls G-fusion in P if and only if $\mathcal{F}_P(G) = \mathcal{F}_P(H)$.

Proof: This is essentially a restatement of the definition of control of G-fusion, and which maps $\phi : A \to B$ lie in the fusion system. The details are left to the reader. $\square$

We have the following corollary of this proposition, our first result about fusion systems proper.

Corollary 1.9 Let G be a finite group and let P be a Sylow p-subgroup. Suppose that P is abelian. Then

$$\mathcal{F}_P(G) = \mathcal{F}_P(\mathrm{N}_G(P)).$$

1.2 Normal p-Complements

One of the first applications of fusion of finite groups was in the question of whether a group has a normal p-complement.

Definition 1.10 Let G be a finite group. Then G is said to have a *normal p-complement* if there exists a subgroup H for which $p \nmid |H|$ and $|G : H|$ is a power of p; i.e., $G = H \rtimes P$, where P is any Sylow p-subgroup of G.

The first results on the question of whether a finite group has a normal p-complement are from Burnside and Frobenius. Burnside's theorem is generally proved as an application of *transfer*, which we will not discuss here (but see, for example, [3, Section 37], [13, Section 7.3], or [22, Chapter 10]).

Frobenius' normal p-complement theorem is a set of three conditions, each equivalent to the presence of a normal p-complement. Modern proofs of this theorem use the machinery of fusion in finite groups, like Grün's first theorem or Alperin's fusion theorem. We will state it now, but not prove it yet, as we do not have Alperin's fusion theorem to hand.

Theorem 1.11 (Frobenius' normal p-complement theorem) Let G be a finite group, and let P be a Sylow p-subgroup of G. Then the following are equivalent:

(i) G possesses a normal p-complement;

(ii) $\mathcal{F}_P(G) = \mathcal{F}_P(P)$;

(iii) every subgroup of the form $\mathrm{N}_G(Q)$ for some non-trivial p-subgroup $Q \leqslant P$ possesses a normal p-complement; and

(iv) for every p-subgroup Q, we have $\mathrm{Aut}_G(Q) = \mathrm{N}_G(Q)/\,\mathrm{C}_G(Q)$ is a p-group.

This is, of course, not exactly what Frobenius proved, but instead of $\mathcal{F}_P(G) = \mathcal{F}_P(P)$ there was a statement about conjugacy in the Sylow p-subgroup, which is easily equivalent.

From this result, we will deduce Burnside's normal p-complement theorem, which is a sufficient, but not necessary, condition to having a p-complement.

Theorem 1.12 (Burnside's normal p-complement theorem) Let G be a finite group, and let P be a Sylow p-subgroup such that $P \leqslant \mathrm{Z}(\mathrm{N}_G(P))$. Then G possesses a normal p-complement.

Proof: Since $P \leqslant \mathrm{Z}(\mathrm{N}_G(P))$, we see that P is abelian. Therefore $\mathcal{F}_P(G) = \mathcal{F}_P(\mathrm{N}_G(P))$ by Corollary 1.9. Furthermore, since P is central in $\mathrm{N}_G(P)$, we see that $\mathcal{F}_P(\mathrm{N}_G(P)) = \mathcal{F}_P(P)$, and so by Frobenius' normal p-complement theorem, G possesses a normal p-complement, as claimed. $\square$

We can quickly derive a result of Cayley from Frobenius' normal p-complement theorem as well, proving that no simple group has a cyclic Sylow 2-subgroup.

Corollary 1.13 (Cayley) Let G be a finite group of even order, and let P be a Sylow p-subgroup of G. If P is cyclic, then G has a normal 2-complement.

Proof: Notice that, if Q is any cyclic 2-group of order 2^m, then $|\operatorname{Aut}(Q)|$ is itself a 2-group. (It is the size of the set

$$\{x \mid 0 < x < 2^m,\ x \text{ is prime to } 2^m\},$$

which has size 2^{m-1}. Thus $\operatorname{Aut}_G(Q)$ is a 2-group for all subgroups $Q \leqslant G$, since Q is cyclic. Thus by Frobenius' normal p-complement theorem, G possesses a normal 2-complement, as claimed. $\square$

Example 1.14 We return to our familiar example, where $G = \mathrm{GL}_3(2)$ and P is the Sylow 2-subgroup considered above. Since $\mathcal{F}_P(G)$ is not $\mathcal{F}_P(P)$, we should have that $\operatorname{Aut}_{\mathcal{F}_P(G)}(Q)$ is not a p-group, for some $Q \leqslant P$. As we saw, the automorphism groups of Q_1 and Q_2, the Klerin four subgroups, have order 6, confirming Frobenius' theorem in this case.

While Frobenius' normal p-complement theorem was a breakthrough, Thompson's normal p-complement theorem was a significant refinement. The original theorem of Thompson [25] proved that, for odd primes, G possesses a normal p-complement if two particular subgroups possess normal p-complements. Glauberman [10] refined this further, proving that, for odd primes, G possesses a normal p-complement if *one* subgroup possesses a normal p-complement! Both Thompson's and Glauberman's results used the 'Thompson subgroup', which we will define now.

Definition 1.15 Let P be a finite p-group, and let $\mathscr{A}$ denote the set of all abelian subgroups of P of maximal order. The *Thompson subgroup*, $J(P)$, is defined to be the subgroup generated by all elements of $\mathscr{A}$.

There are several definitions of the Thompson subgroup in the literature, but this one will do fine for our purposes. We are now in a position to state the theorem.

Theorem 1.16 (Glauberman–Thompson, [10] [13, Theorem 8.3.1]) Let p be an odd prime, and let G be a finite group. Let P be a Sylow p-subgroup of G and write $N = \mathrm{N}_G(\mathrm{Z}(J(P)))$. Then $\mathcal{F}_P(G) = \mathcal{F}_P(P)$ if and only if $\mathcal{F}_P(N) = \mathcal{F}_P(P)$.

It may seem very surprising that a single subgroup controls whether the whole group possesses a normal p-complement, but this is indeed the case. This theorem tells us that, with the notation given there, if $\mathcal{F}_P(N) = \mathcal{F}_P(P)$, then $\mathcal{F}_P(N) = \mathcal{F}_P(G)$. Thus one way of looking at this theorem is that it gives a sufficient condition for N to control G-fusion in P.

In fact, this happens much more often. Glauberman's ZJ-theorem is a sufficient condition for this subgroup N given above to control G-fusion in P. It holds, for each odd prime, for every group that does not involve a particular group, denoted by $Qd(p)$. Let p be a prime, and let $Q = C_p \times C_p$: this can be thought of as a 2-dimensional vector space, and so $\mathrm{SL}_2(p)$ acts on this group in a natural way. Define $Qd(p)$ to be the semidirect product of Q and $\mathrm{SL}_2(p)$.

Example 1.17 In the case where $p = 2$, the group $Qd(p)$ has a normal elementary abelian subgroup of order 4, and is the semidirect product of this group and $\mathrm{SL}_2(2) = S_3$. Hence, $Qd(2) = S_4$, the symmetric group on four letters.

Proposition 1.18 Let G be the group $Qd(p)$, and let P be a Sylow p-subgroup of G. Then $\mathcal{F}_P(G) \neq \mathcal{F}_P(N)$, where $N = \mathrm{N}_G(\mathrm{Z}(J(P)))$.

Proof: The Sylow p-subgroup of $\mathrm{SL}_2(p)$ is cyclic, of order p, and so P is extraspecial of order p^3. It is also easy to see that P has exponent p. Since every subgroup of index p is abelian, the Thompson subgroup of P is all of P, and so $\mathrm{Z}(J(P)) = \mathrm{Z}(P)$. Write Q for the normal subgroup $C_p \times C_p$ in the semidirect product, and N for $\mathrm{N}_G(\mathrm{Z}(P))$. [This is equal to $\mathrm{N}_G(P)$, but we do not need this.]

Since all of $\mathrm{SL}_2(p)$ acts on the subgroup Q, we see that all non-identity elements of this subgroup are conjugate. This cannot be true in N since $\mathrm{Z}(P)$, which has order p and lies inside Q, is normal in this subgroup. Hence $\mathcal{F}_P(G) \neq \mathcal{F}_P(N)$, as claimed. □

Thus if $G = Qd(p)$, then the subgroup N considered above does not control G-fusion in P. The astonishing thing is that $Qd(p)$ is the only obstruction to the statement.

Theorem 1.19 (Glauberman ZJ-theorem) Let p be an odd prime, and let G be a finite group with no subquotient isomorphic with $Qd(p)$ (i.e., G has no subgroup H such that $Qd(p)$ is a quotient of H). Let P be a Sylow p-subgroup, and write $N = \mathrm{N}_G(\mathrm{Z}(J(P)))$. Then $\mathcal{F}_P(N) = \mathcal{F}_P(G)$.

Many of the results given above have analogues for fusion systems. Some are almost direct translations but, for example, Glauberman's ZJ-theorem requires a bit of thought to be converted adequately to fusion systems. The reason for this is that the condition of the theorem – that $Qd(p)$ is not involved in the group – needs to be separated from the language of groups.

1.3 Alperin's Fusion Theorem

Alperin's fusion theorem [1] is one of the fundamental results on fusion in finite groups, and in some sense gives justification to the goal of local finite group theory. A *p-local subgroup* is the normalizer of a (non-trivial) p-subgroup (and sometimes the centralizer of a (non-trivial) p-subgroup as well). One of the main ideas in finite group theory, during the 1960s in particular, is that the structure of p-local subgroups, especially for the prime 2, should determine the global structure of a finite simple group, or more generally an arbitrary finite group, in some sense. We saw an example of such a theorem in Glauberman–Thompson p-nilpotence, which said that whether a finite group G possessed a normal p-complement or not (a *global* property) depends only on what happens in one particular p-local subgroup (a *local* property).

Alperin's fusion theorem is the ultimate justification of this approach, at least in terms of fusion of p-elements, because it tells you that if x and y are two elements of a Sylow p-subgroup P, then you can tell whether x and y are conjugate in G only by looking at p-local subgroups, for various subgroups of P. Example 1.5 shows that fusion in Sylow p-subgroups need not be controlled by any single p-local subgroup, but we proved there that once one took the right collection of p-local subgroups, we could determine conjugacy, by repeatedly conjugating an element inside different p-local subgroups until we reached our target. Alperin's fusion theorem states that this behaviour occurs in *every* finite group. Moreover, the p-local subgroups we need are a very restricted subset.

Definition 1.20 Let G be a finite group, and let P and Q be Sylow p-subgroups of G. We say that $R = P \cap Q$ is a *tame intersection* if both $\mathrm{N}_P(R)$ and $\mathrm{N}_Q(R)$ are Sylow p-subgroups of $\mathrm{N}_G(R)$.

Examples of tame intersections are when the intersection is of index p in one (and hence both) of the Sylow subgroups, and in general if the intersection is normal in both Sylow subgroups. There are, however, other examples.

Theorem 1.21 (Alperin's fusion theorem) Let G be a finite group, and let P be a Sylow p-subgroup of G. Let A and B be two subsets of P such that $A = B^g$. Then there exist Sylow p-subgroups $Q_1, \dots, Q_n$, elements $x_1, \dots, x_n$, and an element $y \in \mathrm{N}_G(P)$ such that

(i) $g = x_1 x_2 \dots x_n y$;

(ii) $P \cap Q_i$ is a tame intersection for all i;

(iii) x_i is a p-element of $\mathrm{N}_G(P \cap Q_i)$ for all i; and

(iv) $A^{x_1 x_2 \dots x_i}$ is a subset of $P \cap Q_{i+1}$ for all $0 \leqslant i \leqslant n-1$.

Proof: For the duration of this proof, fix a Sylow p-subgroup P. We introduce the relation $\to$ on $\mathrm{Syl}_p(G)$. We will show that it is reflexive and transitive, but note that it is *not* symmetric. (In Alperin's original paper [1] and in [13], the symbol $\sim$ was used. We prefer the notation of [3] because the symbol $\sim$ might suggest that the relation is symmetric.) Our ultimate goal is to show that for every $Q \in \mathrm{Syl}_p(G)$, we have that $Q \to P$. We will define this relation now, and prove Alperin's fusion theorem from the claim that $Q \to P$ for all Sylow p-subgroups Q. The definition of $R \to Q$ will be distinctly reminiscent of the theorem itself.

Let Q and R be Sylow p-subgroups of G. We write $R \to Q$ if there exist Sylow p-subgroups $Q_1, \dots, Q_n$ and elements $x_1, \dots, x_n$ such that

(a) $R^{x_1 x_2 \dots x_n} = Q$;

(b) $P \cap Q_i$ is a tame intersection for all i;

(c) x_i is a p-element of $\mathrm{N}_G(P \cap Q_i)$ for all i; and

(d) $(P \cap R)^{x_1 x_2 \dots x_i} \leqslant P \cap Q_{i+1}$ for all $0 \leqslant i \leqslant n-1$.

If we need to specify the element $x = x_1 x_2 \dots x_n$ that is conjugating R to Q, we write $R \to Q$ *via x*.

Suppose that, for all $Q \in \mathrm{Syl}_p(G)$, we have that $Q \to P$. Let A and B be subsets of P with $A^g = B$ for some $g \in G$. Then $B = A^g \leqslant P^g$, and so $A^g \leqslant P \cap P^g$. Hence $A \leqslant P \cap P^{g^{-1}}$. By hypothesis, there is some $x \in G$ such that $P^{g^{-1}} \to P$ via x. This also yields a set $Q_1, \dots, Q_n$ of Sylow p-subgroups and p-elements x_i of $\mathrm{N}_G(P \cap Q_i)$ for all i with $x = x_1 x_2 \dots x_n$. Clearly $x^{-1}g$ lies in $\mathrm{N}_G(P)$ by property (a), and so we take y in the statement of the theorem to be $x^{-1}g$. Then (i) is satisfied by these choices, and (ii) and (iii) are satisfied by properties (b) and (c) respectively. Finally, since $A \subseteq P \cap P^{g^{-1}}$, assertion (iv) from the theorem follows from property (d), as claimed.

We now need to prove that $Q \to P$ for every Sylow p-subgroup Q.

Step 1: $\to$ is reflexive and transitive. It is clearly reflexive as $Q \to Q$ via the identity. If $S \to R$ and $R \to Q$, then there are two collections of Sylow p-subgroups R_i and Q_j, and p-elements $x_i \in \mathrm{N}_G(P \cap R_i)$ and $y_j \in \mathrm{N}_G(P \cap Q_j)$, for $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, such that (writing $x = x_1 x_2 \dots x_n$ and $y = y_1 y_2 \dots y_m$) $S^x = R$ and $R^y = Q$, and for all $0 \leqslant i \leqslant n-1$ and $0 \leqslant j \leqslant m-1$, we have

$$(P \cap S)^{x_1 x_2 \dots x_i} \leqslant P \cap R_{i+1} \text{ and } (P \cap R)^{y_1 y_2 \dots y_j} \leqslant P \cap Q_{j+1}.$$

Then consider the sequence $R_1, \dots, R_n, Q_1, \dots, Q_m$ and the p-elements $x_1, \dots, x_n, y_1, \dots, y_m$, as a candidate pair of sequences for $S \to Q$. Properties (a), (b), and (c) are clear, and property (d) is easy to see. Therefore $\to$ is transitive, as claimed.

Step 2: If $Q, R \in \mathrm{Syl}_p(G)$ such that $P \cap R \geqslant P \cap Q$, $R \to P$ via x and $Q^x \to P$, then $Q \to P$. We prove that in this case, $Q \to Q^x$, since then we are done by Step 1. If $Q_1, \dots, Q_n$ and $x_1, \dots, x_n$ are the two sequences associated with $R \to P$, then the same two sequences prove that $Q \to Q^x$. To see this, note that properties (a), (b), and (c) all hold trivially, and so it remains to show that (d) holds. This property holds since

$$(P \cap Q)^{x_1 x_2 \dots x_i} \leqslant (P \cap R)^{x_1 x_2 \dots x_i} \leqslant P \cap Q_{i+1}$$

for all $0 \leqslant i \leqslant n-1$.

Step 3: Suppose that Q and R are Sylow p-subgroups of G, and that $P \cap Q < R \cap Q$, and $R \to P$ via x. If $S \to P$ for all $S \in \mathrm{Syl}_p(G)$ with $|P \cap S| > |P \cap Q|$, then $Q \to P$. If $Q^x \to P$, then Step 2 would finish the claim if we knew that $P \cap R \geqslant P \cap Q$. However,

$$P \cap R \geqslant P \cap (R \cap Q) \geqslant P \cap (P \cap Q) = P \cap Q,$$

as needed. It remains to prove that $Q^x \to P$. This is just as easy: since $R^x = P$, we have that $P \cap Q^x = (R \cap Q)^x$, and the order of $(R \cap Q)^x$ is equal to $R \cap Q$, which contains $P \cap Q$ properly by assumption. Therefore $Q^x \to P$ by our assumptions, and we have proved the claim.

Step 4: If $Q \in \mathrm{Syl}_p(G)$ with $Q \cap P$ a tame intersection, and for all $R \in \mathrm{Syl}_p(G)$ with $|R \cap P| > |Q \cap P|$ we have that $R \to P$, then $Q \to P$. Firstly, we may assume that $P \neq Q$; since $P \cap Q$ is a tame intersection, then $\bar{P} = \mathrm{N}_P(P \cap Q)$ and $\bar{Q} = \mathrm{N}_Q(P \cap Q)$ are Sylow p-subgroups of $\mathrm{N}_G(P \cap Q)$, and $P \cap Q < \bar{P}$. Write H for the subgroup of $\mathrm{N}_G(P \cap Q)$ generated by all p-elements. Since $\bar{P}$ and $\bar{Q}$ are Sylow p-subgroups of H, there is $x \in H$ such that $\bar{Q}^x = \bar{P}$, and since H is generated by p-elements, we may write $x = x_1 x_2 \dots x_n$, where each x_i is a p-element of $\mathrm{N}_G(P \cap Q)$. Now define Q_i to be Q for each $1 \leqslant i \leqslant n$, and consider the two sequences $Q_1, \dots, Q_n$ and $x_1, \dots, x_n$. We claim that these two sequences yield $Q \to Q^x$. Certainly (a) holds, and $P \cap Q_i = P \cap Q$ is a tame intersection by assumption, so that (b) holds. Since x_i is a p-element of $\mathrm{N}_G(P \cap Q)$, (c) holds as well. Finally,

$$(P \cap Q)^{x_1 \dots x_i} = P \cap Q = P \cap Q_{i+1}$$

for all $0 \leqslant i \leqslant n-1$, so that (d) is satisfied. We also have that $Q^x \to P$, since

$$P \cap Q^x \geqslant P \cap \bar{Q}^x = P \cap \bar{P} = \bar{P} > P \cap Q,$$

and therefore by assumption $Q^x \to P$ since $|P\cap Q^x| > |P\cap Q|$. Thus by transitivity, $Q \to P$, as claimed.

Step 5: For all Sylow p-subgroups Q, we have that $Q \to P$. Let Q be a counterexample to the claim such that $P \cap Q$ is of maximal order. Since $P \to P$ from Step 1, we see that $P \neq Q$, so that $P \cap Q \neq P$; therefore, $P \cap Q < \mathrm{N}_P(P \cap Q)$. Any Sylow p-subgroup of $\mathrm{N}_G(P\cap Q)$ may be written as $\mathrm{N}_R(P\cap Q)$ for some $R \in \mathrm{Syl}_p(G)$ (extend a Sylow p-subgroup of $\mathrm{N}_G(P\cap Q)$ to a Sylow p-subgroup of G) and so let R be such that $\mathrm{N}_P(P\cap Q) \leqslant \mathrm{N}_R(P\cap Q)$ and $\mathrm{N}_R(P\cap Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(P\cap Q)$. By maximality of counterexample, we see that $R \to P$ via some element, say x.

If we can show that $Q^x \to P$, then we are done, since by Step 2, $Q \to P$. We first claim that we must have that $(P\cap Q)^x = P\cap Q^x$. To see this, $(P \cap Q)^x \leqslant R^x = P$, so

$$P\cap Q^x \geqslant P\cap (P\cap Q)^x = (P\cap Q)^x.$$

If $|P\cap Q^x| > |P\cap Q|$, then by choice of counterexample $Q^x \to P$ and we are done. Thus $|P\cap Q^x| = |P\cap Q|$, and our claim is proved.

Let R be a Sylow p-subgroup of G such that

$$\mathrm{N}_{Q^x}(P\cap Q^x) \leqslant \mathrm{N}_S(P\cap Q^x) \in \mathrm{Syl}_p(\mathrm{N}_G(P\cap Q^x)).$$

As before, $P\cap Q^x < \mathrm{N}_{Q^x}(P\cap Q^x) \leqslant S$, so again as before, we see that $P\cap Q^x$ is properly contained within $S\cap Q^x$. Applying Step 3, we need only that $S \to P$ to have that $Q^x \to P$, which we have already observed leads to $Q \to P$. Therefore, by maximality of $|P\cap Q|$, we have that

$$P\cap Q^x = P\cap S.$$

The final step is to claim that $P\cap S$ is a tame intersection. In that case, Step 4 proves that $S \to P$, resulting in a final contradiction. By choice of S, we have that $\mathrm{N}_S(P\cap Q^x)$ is a Sylow p-subgroup of $\mathrm{N}_G(P\cap Q^x)$, and since $P\cap Q^x = P\cap S$, we have one half of a tame intersection. By our choice of R, we have that $\mathrm{N}_R(P\cap Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(P\cap Q)$, and so we may 'conjugate' this statement by x (with recalling that $(P\cap Q)^x = P\cap Q^x = P\cap S$) to get

$$\mathrm{N}_{R^x}(P\cap S) \in \mathrm{Syl}_p(\mathrm{N}_G(P\cap S)).$$

Since $R^x = P$, we get the other condition for $P\cap Q$ to be a tame intersection, and the proof is complete. □

In [1], Alperin goes on to show that if one relaxes statement (i) in the theorem to simply '$A^{x_1\dots x_n y} = B$', then one may impose the extra condition that, writing $R = P\cap Q$, we have that $\mathrm{C}_P(R) \leqslant R$.

Definition 1.22 Let G be a finite group and let P be a Sylow p-subgroup of G.

(i) A *family* is a collection of pairs (Q, X), where Q is a subgroup of P and X is a subset of $\mathrm{N}_G(Q)$.

(ii) A family F is called a *weak conjugation family* if, whenever A and B are subsets of P with $A^g = B$ for some $g \in G$, there exist elements $(Q_1, X_1), (Q_2, X_2), \dots, (Q_n, X_n)$ of F and elements $x_1, \dots, x_n, y$ of G such that

  (a) $A^{x_1 x_2 \dots x_n y} = B$;

  (b) x_i is an element of X_i for all i and $y \in \mathrm{N}_G(P)$; and

  (c) $A^{x_1 x_2 \dots x_i} \subseteq Q_{i+1}$ for all $0 \leqslant i \leqslant n-1$.

(iii) A weak conjugation family F is called a *conjugation family* if, in addition, we have $x_1 \dots x_n y = g$ for some choice of the (Q_i, X_i), x_i, and y.

Alperin's fusion theorem states that if F_t is the family (R, X), where $R = P \cap Q$ is a tame intersection of P and $Q \in \mathrm{Syl}_p(G)$ and X is the set of p-elements of $\mathrm{N}_G(R)$, then F_t is a conjugation family. Let F_c denote the subset of F_t consisting only of pairs (R, X) such that $\mathrm{C}_P(R) \leqslant R$. Then Alperin proves the following theorem in [1].

Theorem 1.23 (Alperin [1]) The family F_c given above is a weak conjugation family.

Goldschmidt [12] examined Alperin's proof more closely, and proved that a refinement of the theorem was possible, further reducing the subgroups needed. To state this restriction, we first need the definition of a strongly p-embedded subgroup. Let G be a finite group with $p \mid |G|$, and let M be a subgroup of G. We say that M is *strongly p-embedded* if M contains a Sylow p-subgroup of G, and $M \cap M^g$ is a p'-group for all $g \in G \setminus M$.

Theorem 1.24 (Goldschmidt [12]) Let G be a finite group and let P be a Sylow p-subgroup of G. Let F denote the family of all pairs $(R, \mathrm{N}_G(R))$, where R is a subgroup of P for which the following four conditions hold:

(i) R is a tame intersection $P \cap Q$, where $Q \in \mathrm{Syl}_p(G)$;

(ii) $\mathrm{C}_P(R) \leqslant R$;

(iii) R is a Sylow p-subgroup of $\mathrm{O}_{p',p}(\mathrm{N}_G(R))$; and

(iv) $R = P$ or R has a strongly p-embedded subgroup.

Then F is a weak conjugation family.

We will not prove either of Theorems 1.23 or 1.24 here. The latter theorem has an analogue for fusion systems in general (Theorem 3.21), as we shall see in Section 3.4.

1.4 Fusion Systems

Having defined a fusion system of a finite group, we now turn to defining a fusion system in general. Like that of finite groups, this takes place over a finite p-group, and like that of finite groups, it involves injective homomorphisms between subgroups of groups. Since we have no underlying group from which to draw our morphism sets, we need to make some compatibility conditions on the morphisms.

Definition 1.25 Let P be a finite p-group. Then a *fusion system* $\mathcal{F}$ on P is a category, whose objects are all subgroups of P, and whose morphisms $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ are subsets of all injective homomorphisms $Q \to R$, where Q and R are subgroups of P, with composition of morphisms given by the usual composition of homomorphisms. The sets $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ should satisfy the following three axioms:

(i) for each $g \in P$ with $Q^g \leqslant R$, the associated conjugation map $\theta_g : Q \to R$ is in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$;

(ii) for each $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, the isomorphism $Q \to Q\phi$ lies in $\mathrm{Hom}_{\mathcal{F}}(Q,Q\phi)$; and

(iii) if $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ is an isomorphism, then its inverse $\phi^{-1} : R \to Q$ lies in $\mathrm{Hom}_{\mathcal{F}}(R,Q)$.

We will unravel the definition slightly now: the first condition requires that all morphisms in $\mathcal{F}_P(P)$ lie in $\mathcal{F}$; the second condition says that if one map ϕ with domain Q is in the fusion system then so is the induced isomorphism $\bar{\phi} : Q \to \operatorname{im}\phi$; and the final axiom requires that $\mathcal{F}$-isomorphism is an equivalence relation.

The next proposition is clear, and its proof is left as an exercise.

Proposition 1.26 Let G be a finite group and let P be a Sylow p-subgroup. Then $\mathcal{F}_P(G)$ is a fusion system on P.

The concept of a fusion system is a little loose for good theorems to be proved about it, and we prefer to deal with *saturated* fusion systems. To define a saturated fusion system, we need to define the concept of fully centralized and fully normalized subgroups.

Definition 1.27 Let P be a finite p-group, and let Q be a subgroup of P. Let $\mathcal{F}$ be a fusion system on P.

(i) The subgroup Q is said to be *fully centralized* if, whenever $\phi : Q \to R$ is an isomorphism in $\mathcal{F}$, we have that
$$|\,\mathrm{C}_P(Q)| \geqslant |\,\mathrm{C}_P(R)|.$$

(ii) The subgroup Q is said to be *fully normalized* if, whenever $\phi : Q \to R$ is an isomorphism in $\mathcal{F}$, we have that

$$|\,\mathrm{N}_P(Q)| \geqslant |\,\mathrm{N}_P(R)|.$$

Write $\mathcal{F}^{\mathrm{f}}$ for the set of all fully normalized subgroups of P.

We now come to the definition of a saturated fusion system. This definition appears a bit convoluted, and we will try to motivate it afterwards.

Definition 1.28 Let P be a finite p-group, and let $\mathcal{F}$ be a fusion system on P. We say that $\mathcal{F}$ is *saturated* if

(i) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$, and

(ii) every morphism $\phi : Q \to P$ in $\mathcal{F}$ such that $Q\phi$ is fully normalized extends to a morphism $\bar{\phi} : N_\phi \to P$, where

$$N_\phi = \{x \in \mathrm{N}_P(Q) \,:\, \text{there exists } y \in \mathrm{N}_P(Q\phi) \text{ such that } (g^x)\phi = (g\phi)^y \text{ for all } g \in Q\}.$$

We need to motivate the definition of N_ϕ. Let ϕ be a map from Q to P. There is an induced map $\phi' : \mathrm{Aut}_P(Q) \to \mathrm{Aut}(Q\phi)$, such that

$$\theta_g \mapsto \phi^{-1}\theta_g\phi.$$

We would like the image of ϕ' to be a subgroup of $\mathrm{Aut}_P(Q\phi)$, but in general this won't be true, and so we consider the preimage of $\mathrm{Aut}_P(Q\phi)$ under this map ϕ'. This is some subgroup X of $\mathrm{Aut}_P(Q)$, and this has a corresponding subgroup Y in $\mathrm{N}_P(Q)$ containing $\mathrm{C}_P(Q)$, since

$$\mathrm{Aut}_P(Q) \cong \mathrm{N}_P(Q)/\,\mathrm{C}_P(Q)$$

through the standard isomorphism taking $g \in \mathrm{N}_P(Q)$ to θ_g. This subgroup Y is exactly the subgroup N_ϕ, defined above. Thus the subgroup N_ϕ is the largest subgroup of $\mathrm{N}_P(Q)$ such that $(N_\phi/\,\mathrm{C}_P(Q))^{\phi'} \leqslant \mathrm{Aut}_P(Q\phi)$. This is an attempt to give an idea as to why the subgroup N_ϕ is considered, and this alternative viewpoint be useful at several points in the sequel.

Saturated fusion systems have a lot more structure, and are a lot closer to the fusion systems arising from finite groups. We will prove that every fusion system arising from a finite group is saturated, but before we do that, we will need a characterization of fully normalized subgroups for fusion systems of finite groups.

Proposition 1.29 Let G be a finite group, and let P be a Sylow p-subgroup of G. Let Q be a subgroup of P. Then $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$ if and only if $|\,\mathrm{N}_P(Q)| \geqslant |\,\mathrm{N}_P(Q^g)|$ for all $g \in G$.

Proof: Let R be a Sylow p-subgroup of $\mathrm{N}_G(Q)$ containing $\mathrm{N}_P(Q)$. Thus there is an element $g \in G$ such that $R^g \leqslant P$, and so $R^g \leqslant \mathrm{N}_P(Q^g)$. Thus

$$|\,\mathrm{N}_P(Q)| \leqslant |R| \leqslant |\,\mathrm{N}_P(Q^g)|.$$

If $|\,\mathrm{N}_P(Q)| \geqslant |\,\mathrm{N}_P(Q^g)|$ for all $g \in G$, then $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$. Conversely, if $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$, then $|\,\mathrm{N}_P(Q)| = |R|$, and this the order of a Sylow p-subgroup of $\mathrm{N}_G(Q^g) = \mathrm{N}_G(Q)^g$. Hence we get the result. $\square$

Note that a similar result holds for centralizers, and the proof is very similar.

Theorem 1.30 Let G be a finite group and let P be a Sylow p-subgroup of G. Then $\mathcal{F}_P(G)$ is a saturated fusion system.

Proof: Since $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{N}_G(P)/\,\mathrm{C}_G(P)$, and the image of P in this quotient group is a Sylow p-subgroup, the first axiom is satisfied. Thus, let Q be a subgroup of P and let $\phi : Q \to P$ be a morphism in $\mathcal{F}$, and suppose that $Q\phi$ is fully normalized. Since ϕ is induced by conjugation, there is some $g \in G$ such that $x\phi = x^g$ for all $x \in Q$. In this case, the set N_ϕ is given by

$$N_\phi = \{x \in \mathrm{N}_P(Q) \,:\, \text{there exists } y \in \mathrm{N}_P(Q^g) \text{ such that } z^{xg} = z^{gy} \text{ for all } z \in Q\}.$$

Thus $x \in N_\phi$ if and only if $xgy^{-1}g^{-1}$ centralizes Q. Then $g^{-1}xgy^{-1}$ centralizes Q^g, and so $x^g = hy$, for some $h \in \mathrm{C}_G(Q^g)$. Thus

$$(N_\phi)^g \leqslant \mathrm{N}_P(Q^g)\,\mathrm{C}_G(Q^g).$$

Since N_ϕ is a p-subgroup, and by Proposition 1.29, $\mathrm{N}_P(Q^g)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q^g)$, there is some element a of $\mathrm{C}_G(Q^g)$ such that $(N_\phi)^{ga}$ is contained within $\mathrm{N}_P(Q^g)$.

Define $\theta : N_\phi \to P$ by $x\theta = x^{ga}$, for all $x \in N_\phi$. Since $a \in \mathrm{C}_G(P^g)$, θ extends ϕ, and so this is the map required by the definition. $\square$

We end the section with a discussion of the so-called Solomon fusion system, which is one of the foundations of the subject. Let G be the group $\mathrm{Spin}_7(3)$, and let $H = G/\langle z\rangle$, where z is the central involution; then H has a Sylow 2-subgroup isomorphic with that of A_{12}. Solomon proved the following.

Theorem 1.31 (Solomon) There does not exist a finite group K with the following properties:

(i) a Sylow 2-subgroup P of K is isomorphic with that of $\mathrm{Spin}_7(3)$;

(ii) $\mathcal{F}_P(\mathrm{Spin}_7(3)) \subseteq \mathcal{F}_P(K)$; and

(iii) all involutions in K are conjugate.

Theorems such as these are often proved using local analysis on the 2-local structure of the group. Solomon attempted this, but found that no contradiction could be reached this way; he was forced to find another way. The reason for this is the following.

Theorem 1.32 Let P be isomorphic with the Sylow 2-subgroup of $\mathrm{Spin}_7(3)$. Then there exists a saturated fusion system $\mathcal{F}$ on P such that $\mathcal{F}_P(\mathrm{Spin}_7(3)) \subseteq \mathcal{F}$ and all involutions are $\mathcal{F}$-isomorphic.

This fusion system is an example of an *exotic* fusion system. The interesting thing about this is that the fusion system is also 'simple', a term that will be defined later in the course.

Let q be an odd prime power such that $q \equiv \pm 3 \mod 8$, and let P be a Sylow 2-subgroup of $\mathrm{Spin}_7(q)$. Solomon actually showed that there does not exist a finite group having P as a Sylow 2-subgroup, with a single conjugacy class of involutions, and such that another technical condition on centralizers holds, that we will examine later.

Levi and Oliver [17] proved that for all odd q, there exists a saturated fusion system on P that has the above properties. Furthermore, they are examples of a special type of fusion system called a *simple* fusion system. (We will see the definition of a simple fusion system later, along with the definition of normal fusion systems.) They are the only known examples of simple fusion systems that do not arise from finite groups.

1.5 Frobenius' Normal p-Complement Theorem

We end this chapter with a section on the proof of Frobenius' normal p-complement theorem. Let G be a finite group and let P be a Sylow p-subgroup of G. Recall that Frobenius' theorem states that the following are equivalent:

(i) G possesses a normal p-complement, so that there is a subgroup H such that $G = H \rtimes P$;

(ii) $\mathcal{F}_P(G) = \mathcal{F}_P(P)$;

(iii) every subgroup of the form $\mathrm{N}_G(Q)$ for some non-trivial p-subgroup $Q \leqslant P$ possesses a normal p-complement; and

(iv) for every p-subgroup Q, we have $\mathrm{Aut}_G(Q) = \mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ is a p-group.

Some of these implications are obvious, but one in particular requires a lot of work to do. We firstly prove that (i) implies (ii).

Proposition 1.33 Let G be the semidirect product of H by P, where $P \in \mathrm{Syl}_p(G)$. Then $\mathcal{F}_P(G) = \mathcal{F}_P(P)$.

Proof: Let A and B be subgroups of P, and suppose that there is some $g \in G$ such that $\theta_g : A \to B$ is a map induced by conjugation by g. Let h be an element of H, and let x be an element of P. Then

$$x^h = h^{-1}xh = h^{-1}xhx^{-1}x = (h^{-1}h')x,$$

which lies in P precisely when $h = h'$, in which case h centralizes x. Thus if $h \in H$ maps A to B, then $h \in \mathrm{C}_G(A)$, and θ_h is trivial on A.

If $g \in G$, then $g = yh$ for some $y \in P$ and $h \in H$, whence A^g is $(A^y)^h$, and since A^y is a subgroup of P, as is $(A^y)^h$, we must have that θ_h centralizes A^y, and so $\theta_g = \theta_y$, as required. □

Proving that (ii) implies (iv) is easy, but we will go via a definition.

Definition 1.34 Let G be a finite group, and $P \in \mathrm{Syl}_p(G)$. Write $\mathcal{F} = \mathcal{F}_P(G)$. If $Q \leqslant P$, then the group $\mathrm{Aut}_{\mathcal{F}}(Q)$ is the collection of all morphisms

$$\mathrm{Hom}_{\mathcal{F}}(Q, Q) = \mathrm{N}_G(Q)/\,\mathrm{C}_G(Q).$$

(This is often called the *automizer* of Q.)

Lemma 1.35 Suppose that $\mathcal{F}_P(G) = \mathcal{F}_P(P)$. Then the automizer of any p-subgroup of G is itself a p-group.

Proof: Suppose that $\mathcal{F} = \mathcal{F}_P(G) = \mathcal{F}_P(P)$; let ϕ be an element of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Since every element of $\mathrm{Aut}_{\mathcal{F}}(Q)$ is induced by conjugation by an element of P, we must have that θ_g has order a power of p, and so $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a p-group, as required. □

Next, we do (i) implies (iii). Recall that a group is called *p-nilpotent* if

$$G = \mathrm{O}_{p'}(G) \rtimes P,$$

where $P \in \mathrm{Syl}_p(G)$. In this case, it is easy to see that $\mathrm{O}_{p'}(G)$ consists of all p'-elements of G.

Lemma 1.36 Suppose that G is a p-nilpotent group, and let H be a subgroup of G. Then H is p-nilpotent.

Proof: Let $K = \mathrm{O}_{p'}(G)$. Since K is the set of all p'-elements of G, we must have that $H \cap K = \mathrm{O}_{p'}(H)$. Since G/K is a p-group, and

$$G/K \cong H/H \cap K,$$

we see that $H/\mathrm{O}_{p'}(H)$ is also a p-group, as required. $\square$

The last but one of the implications is fairly straightforward, namely (iii) implies (iv).

Lemma 1.37 Let G be a finite group and let Q be a non-trivial p-subgroup of G. Suppose that $\mathrm{N}_G(Q)$ is p-nilpotent. Then $\mathrm{Aut}_G(Q)$ is a p-group.

Proof: Certainly Q and $K = \mathrm{O}_{p'}(\mathrm{N}_G(Q))$ are of coprime orders, and are normal subgroups of $\mathrm{N}_G(Q)$. Hence they centralize one another, and so $K \leqslant \mathrm{C}_G(Q)$. Therefore

$$K \leqslant \mathrm{C}_G(Q) \leqslant \mathrm{N}_G(Q),$$

and since $\mathrm{N}_G(Q)/K$ is a p-group, we see that $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ is a p-group, as required. $\square$

It remains to show that (iv) implies (i). For this, we proceed by induction on $|G|$, and so we may assume that every proper subgroup of G possesses a normal p-complement. Let Q be a non-trivial subgroup of P.

If Q is normal in G, then G/Q possess a normal p-complement M/Q, where $M \geqslant Q$. Then Q is a normal Sylow p-subgroup of M, and so $M = Q \rtimes K$ for some p'-subgroup K (by Schur–Zassenhaus). Since $\mathrm{N}_G(Q)/\mathrm{C}_G(Q)$ is a p-group, we must have that $M = Q \times K$. Then K is a normal p-complement in G, as claimed. Thus $\mathrm{O}_p(G) = 1$, and so Q is not normal in G.

The subgroup $\mathrm{N}_G(Q)$ has a normal p-complement by induction, and so $\mathrm{N}_G(Q) = Q \times K$ for some p'-group K. From this, using Alperin's fusion theorem (Theorem 3.21) or an easy direct calculation, we see that $\mathcal{F}_P(G) = \mathcal{F}_P(P)$. In particular, if $\mathrm{Z}(P)^g \leqslant P$, then $\mathrm{Z}(P)^g = \mathrm{Z}(P)$. Set $N = \mathrm{N}_G(\mathrm{Z}(P))$. Since N possesses a normal p-complement, $p \mid |N/N'|$.

Now we quote a theorem of Grün (see [13, Theorem 7.5.2]), which states that $p \mid |N/N'|$ if and only if $p \mid |G/G'|$. Thus $\mathrm{O}^p(G) < G$, and so $\mathrm{O}^p(G)$, and hence G, possess normal p-complements, as required.

Chapter 2

Representation Theory

To each p-block of a finite group, one may associate a fusion system. In this chapter, we will see how to do this, and briefly look at some more advanced topics in this area. We begin by defining blocks and the Brauer morphism, then deal with Brauer pairs, the basis for the definition of a block fusion system. We give some examples of blocks and define the block fusion system at the end of this chapter.

2.1 Blocks and the Brauer morphism

Let G be a finite group, and let k be a field of characteristic p, where $p \mid |G|$. The group algebra kG is no longer semi-simple, unlike the case where $p = 0$ or $p \nmid |G|$, but we may recover something. In the original (complex field) case, we have

$$\mathscr{C}G = \bigoplus_{i=1}^{r} M_{n_i}(\mathscr{C}),$$

by standard Artin–Wedderburn theory. In the case of characteristic p, we have something more complicated.

Definition 2.1 Let k be a field of characteristic p and G be a finite group. A *p-block* (often simply 'block') of the group algebra kG is a two-sided ideal B of kG such that, whenever B can be written as the direct sum $B_1 \oplus B_2$ of two other two-sided ideals of KG, then exactly one of the B_i is zero.

It is clear that there is a decomposition of kG as a sum of blocks, since kG is finite-dimensional. Suppose that B is a block of kG, and write

$$kG = \bigoplus_{i} B_i$$

for some decomposition of kG into blocks. Then, intersecting with B, we get

$$B = B \cap kG = \bigoplus_i (B \cap B_i),$$

and since B is indecomposable, all but one of the $B \cap B_i$ must be zero, and therefore $B = B \cap B_j = B_j$ by indecomposability of B_j. Hence the decomposition of kG into blocks is unique, up to ordering of the factors.

Note that the decomposition of the group algebra into blocks remains valid if k is replaced by any Noetherian ring R.

Theorem 2.2 (Maschke's Theorem) Let G be a finite group and let K be a field of characteristic 0, or of characteristic p where $p \nmid |G|$. Then all blocks are matrix algebras $M_n(\mathscr{C})$ for various n, and KG is semi-simple.

Thus the case of $p \mid |G|$ is the only interesting case, at least from a block-theoretic point of view. Indeed, if $p \mid |G|$, then the algebra kG is not semi-simple, and so not all blocks are matrix algebras.

We now introduce a closely related concept.

Definition 2.3 Let G be a finite group and let R be a Noetherian ring. A *central idempotent* of RG is an idempotent e (i.e., $e \neq 0$ and $e^2 = e$) such that $e \in \mathrm{Z}(RG)$). The central idempotent e is called *primitive* if, whenever $e = e_1 + e_2$ where e_1 and e_2 are central idempotents with $e_1 e_2 = 0$, then $e_1 = 0$ or $e_2 = 0$.

Let e be a central idempotent. Then $RGe = eRG$ (as e is central), and is therefore a two-sided ideal of RG.

Proposition 2.4 Suppose that e is a central idempotent, of the group ring RG. Then e is primitive if and only if RGe is a block.

Proof: Suppose that e is primitive, and write $RGe = B \oplus B'$. If e lies in either B or B' then this decomposition is trivial, and so we will assume that $e = f + f'$. Then, since e is central, we see that f and f' must commute with all of B and B' respectively, and therefore f and f' are central. Also, since $B \cap B' = 0$, we see that $ff' = 0$. Finally,

$$e = e^2 = (f + f')^2 = f^2 + f'^2,$$

and therefore f and f' are both idempotents. Therefore either f or f' is zero, and we get a contradiction.

Now suppose that e is not primitive, and write $e = f + f'$. Then $RGe = RGf + RGf'$, and we have that RGf and RGf' are two-sided ideals. The intersection of $RGf = fRG$ and

RGf' is $fRGf' = RG(ff') = 0$, and so this sum is direct. Therefore RGe is not a block, as required. $\square$

Thus to each block we may associate a block idempotent, and there is a corresponding decomposition of 1 into primitive central idempotents.

We now pause to introduce the concept of a p-modular system. Let k denote a field of characteristic p, and let $\mathcal{O}$ denote a local PID, whose quotient $\mathcal{O}/J(\mathcal{O})$ is isomorphic with the field k. Finally, let K denote the field of fractions of $\mathcal{O}$. Then $(K, \mathcal{O}, k)$ forms a p-modular system. We will assume that

(i) K contains a primitive $|G|$th root of unity and has characteristic 0;

(ii) $\mathcal{O}$ is a complete local ring with respect to the $J(\mathcal{O})$-adic topology; and

(iii) k is algebraically closed.

The assumptions about K will mean that the K-representations of G are the same as the $\mathscr{C}$-representations of G. If e is a central idempotent of $\mathcal{O}G$, then $e + J(\mathcal{O}G)$ is a central idempotent of kG. Clearly if $e + J(\mathcal{O}G)$ is primitive, so is e, and so the blocks of $\mathcal{O}G$ are unions of blocks of kG. In fact, it can be shown that they are the same. However, since the blocks of KG are simply matrix algebras, not all blocks of $\mathcal{O}G$ lift to blocks of KG, but rather to sums of blocks.

Let F be one of k and K, and let G be a finite group. Let M be an indecomposable (right) FG-module. Then $M \cdot 1 = M$. Now let

$$1 = e_1 + e_2 + \cdots + e_r$$

denote a decomposition of 1 into block idempotents. Then

$$M = M \cdot 1 = M \cdot e_1 \oplus M \cdot e_2 \oplus \cdots \oplus M \cdot e_r$$

is a direct decomposition of M, and so all but one of the $M \cdot e_i$ is the zero module. We say that M *belongs* to the block FGe_i if $M \cdot e_i \neq 0$.

Thus this determines a decomposition of the set of simple kG-modules and the simple KG-modules into the various p-blocks of kG, via the correspondence with block idempotents between kG, $\mathcal{O}G$, and KG. The block to which the trivial kG- or KG-module belongs is called the *principal block*.

To end this section, we will define the Brauer morphism, and give one of of its most important propeties. Put simply, the Brauer morphism is a restriction map. Let P be a p-subgroup of G; the *Brauer morphism* $\mathrm{Br}_P : kG \to k\,\mathrm{C}_G(P)$ is the surjective k-linear map

$$\sum_{g \in G} \alpha_g g \mapsto \sum_{g \in \mathrm{C}_G(P)} \alpha_g g.$$

Proposition 2.5 Let P be a p-subgroup of the finite group G. Then Br_P is multiplicative when restricted to the P-stable elements $(kG)^P$ of kG.

Proof: A k-basis for $(kG)^P$ is the set of all P-class sums of elements of G. Let $\mathscr{X}$ denote the set of all P-conjugacy classes of G, and if $X \in \mathscr{X}$, let $\hat{X}$ denote its class sum. Let g be an element of $\mathrm{C}_G(P)$, and consider the multiplicity of g in the product set $X \cdot Y$, where X and Y are in $\mathscr{X}$. Then we claim that either this is 0, or $|X| = |Y|$ and this is $|X|$.

Suppose that this claim is true. Since conjugacy classes have size either 1 or a multiple of p, we see that $\mathrm{Br}_P(\hat{X})$ is $\hat{X}$ if X is a singleton set (lying inside $\mathrm{C}_G(P)$) and 0 otherwise. Thus $\mathrm{Br}_P(\hat{X})\,\mathrm{Br}_P(\hat{Y})$ is 0 unless both X and Y are singleton sets, in which case it is $\hat{X}\hat{Y}$. Conversely, $\mathrm{Br}_P(\hat{X}\hat{Y})$ is $\hat{X}\hat{Y}$ if both X and Y are singleton sets, and is 0 if at least one of them is not, by the claim. Hence Br_P is multiplicative on the basis elements of $(kG)^P$, and so by linearity is multiplicative.

It remains to prove the claim. Let $g \in \mathrm{C}_G(P)$, and suppose that g appears in the product set $X \cdot Y$; write n for the number of distinct ways of making g from one element of X and one from Y. If $|X| = |Y| = 1$ then certainly $n = 1$, and so our claim is true in this case. If $g = xy$, then

$$g = g^h = x^h y^h$$

for every $h \in P$, and since as h runs over all elements of P, all elements of X and all elements of Y appear. Thus there is one way of producing g for every $x \in X$, and one way for every $y \in Y$. Thus $|X| = |Y|$, and there are $|X|$ possible ways. □

The Brauer morphism therefore is a surjective algebra homomorphism

$$\mathrm{Br}_P : (kG)^P \to k\,\mathrm{C}_G(P),$$

for any p-subgroup P.

2.2 Brauer Pairs

A Brauer pair is a very powerful concept, and the basis of our block fusion systems. However, it is also quite easy. The idea is that a Brauer pair is a p-subgroup of G, together with a p-block idempotent of its centralizer.

Definition 2.6 Let G be a finite group and let p be a prime dividing $|G|$. Then a *Brauer pair* is an ordered pair (Q, e), where Q is a p-subgroup of G and e is a primitive central idempotent of $k\,\mathrm{C}_G(Q)$. Denote by $\mathcal{B}(Q)$ the set of block idempotents of $k\,\mathrm{C}_G(Q)$.

Since G acts by conjugation on the set of all p-subgroups and (transporting from Q to Q^g) on the set of all primitive central idempotents of $k\,\mathrm{C}_G(Q)$, we see that G acts by conjugation on the set of all Brauer pairs. Denote by $\mathrm{N}_G(Q,e)$ the set of elements that stabilize the Brauer pair (Q,e) under the conjugation action.

Lemma 2.7 Let G be a finite group, and let R be a p-subgroup of G. Let e be a primitive central idempotent of $\mathrm{C}_G(R)$. Suppose that Q is normal in R. Then there is a unique R-stable block idempotent f of $\mathrm{C}_G(Q)$ such that

$$\mathrm{Br}_R(f)e = e.$$

If f' is a different R-stable block idempotent of Q, then $\mathrm{Br}_R(f')e = 0$.

Proof: Suppose that f is an R-stable block idempotent of $k\,\mathrm{C}_G(Q)$; i.e., suppose that $f \in \mathcal{B}(Q)^R$. Then $f \in (kG)^R$, and so $\mathrm{Br}_R(f)$ is also an idempotent or zero. Since $\mathrm{C}_G(R) \leqslant \mathrm{C}_G(Q)$, we must have that $\mathrm{Br}_R(f)$ is central as f is. Thus either $\mathrm{Br}_R(f)$ is zero or it is a central idempotent of $k\,\mathrm{C}_G(R)$.

Since R acts by conjugation on $\mathcal{B}(Q)$, the R-orbits that are not fixed points have length a multiple of p, and are hence zero as k has characteristic p. Thus

$$1 = \mathrm{Br}_R(1) = \sum_{b\in\mathcal{B}(Q)^R} \mathrm{Br}_R(b),$$

whence one of the R-stable block idempotents of $k\,\mathrm{C}_G(Q)$ has non-zero image under the Brauer morphism, which we may choose to be f. Also, since $ff' = 0$ for any element $f' \in \mathcal{B}(Q)^R$, we see that $\mathrm{Br}_R(f)\,\mathrm{Br}_R(f') = 0$. Thus

$$e = \sum_{b\in\mathcal{B}(Q)^R} \mathrm{Br}_R(b)e,$$

and since e is primitive, exactly one of the $\mathrm{Br}_R(b)e$ is non-zero, as required. □

This allows us to define a partial order relation on the set of all Brauer pairs.

Definition 2.8 Let (Q,f) and (R,e) be Brauer pairs.

(i) Define $(Q,f) \trianglelefteqslant (R,e)$ if $Q \trianglelefteqslant R$, the block idempotent f is R-stable, and $\mathrm{Br}_R(f)e = e$.

(ii) Define $\leqslant$ to be the transitive extension of $\trianglelefteqslant$.

The relation $\leqslant$ on Brauer pairs is clearly reflexive, and anti-symmetric, and by definition transitive. By Lemma 2.7, given $Q \leqslant R$ and a block idempotent $e \in \mathcal{B}(R)$, there is some Brauer pair (Q,f) such that $(Q,f) \leqslant (R,e)$. We would like this to be unique; such a statement would follow from the case where $Q \trianglelefteqslant R$.

Lemma 2.9 Let R be a p-subgroup of G, and suppose that P and Q are normal subgroups of R such that $P \leqslant Q$. Let e be a block idempotent of $k\,\mathrm{C}_G(R)$. Write f_1 and f_2 for the (unique) elements of $\mathcal{B}(P)^R$ and $\mathcal{B}(Q)^R$ such that $\mathrm{Br}_R(f_i)e = e$. Write f for the (unique) element of $\mathcal{B}(P)^Q$ such that $\mathrm{Br}_Q(f)f_2 = f_2$. Then $f = f_1$.

Proof: We will show that f is R-stable, and that $\mathrm{Br}_R(f)e = e$. Let x be an element of R; then $f^x \in \mathcal{B}(P)$. We know that f is Q-stable, and so f^x is Q-stable also (as $Q \trianglelefteq R$). Thus $\mathrm{Br}_Q(f^x) = \mathrm{Br}_Q(f)^x$. However, f_2 is R-stable, and so

$$\mathrm{Br}_Q(f^x)f_2 = \mathrm{Br}_Q(f)^x f_2^x = f_2^x = f_2,$$

and so $f^x = f$, proving R-stability (via Lemma 2.7).

For the second claim, note that

$$\mathrm{Br}_R(f)e = \mathrm{Br}_R(f)\,\mathrm{Br}_R(f_2)e = \mathrm{Br}_R(\mathrm{Br}_Q(f)f_2)e = \mathrm{Br}_R(f_2)e = e.$$

Therefore, by Lemma 2.7, we see that $f = f_1$. □

What this essentially says is that if $(P, f) \trianglelefteq (Q, f')$ and $(Q, f') \trianglelefteq (R, e)$ then $(P, f) \trianglelefteq (R, e)$, where P, Q, and R are as in the lemma. Therefore given a $Q \leqslant R$ and a block idempotent $e \in \mathcal{B}(R)$, there is a unique $f \in \mathcal{B}(Q)$ such that $(Q, f) \leqslant (R, e)$.

Having built up the machinery of Brauer pairs, we are in a position to start developing the structure of the block fusion system. This starts with the notion of a b-Brauer pair.

Definition 2.10 Let b be a block of kG. Then a *b-Brauer pair* is a Brauer pair (R, e) such that

$$(1, b) \leqslant (R, e).$$

A *maximal* b-Brauer pair is a b-Brauer pair (D, e) such that $|D|$ is maximal. The subgroup D is called a *defect group* of the block b.

Notice that if b is a block of kG then b is fixed under conjugation (as b is central) and so G acts on the set of all b-Brauer pairs by conjugation. In particular, if D is a defect group of b then so is D^g, and so the defect groups of a block are unions of conjugacy classes of p-subgroups of G.

In the statement of the next theorem, we need the concept of a relative trace: put simply, if x is H-stable, then

$$\mathrm{Tr}_H^G(x) = \sum_{g \in T} xg,$$

where T is a right transversal to H in G. If x belongs to some H-stable space X, then $\mathrm{Tr}_H^G(X)$ denotes the image of the relative trace map inside X^G.

Theorem 2.11 Let b be a block idempotent of kG. Then the minimal such p-subgroup P such that

$$b \in \mathrm{Tr}_P^G(kGb)^P$$

is a defect group of b. Furthermore, G acts transitively on the set of all defect groups of b.

This theorem will not be proved here. Note that the defect groups of a block therefore form a conjugacy class of p-subgroups of G. In fact, we can do better than that.

Theorem 2.12 Let b be a block idempotent of kG. Then G acts transitively on the set of maximal b-Brauer pairs, and if (D, e) is such a pair, then $\mathrm{N}_G(D, e)/D\,\mathrm{C}_G(D)$ is a p'-group.

2.3 Block Fusion Systems

Definition 2.13 Let G be a finite group and let k be a field of characteristic p. Let b be a block idempotent of kG, and (D, e_D) denote a maximal b-Brauer pair. Denote by $\mathcal{F} = \mathcal{F}_{(D,e_D)}(G, b)$ the category whose objects are all subgroups of D, and whose morphisms sets are described below. Let Q and R be subgroups of D, and e_Q and e_R be the unique block idempotents such that $(Q, e_Q) \leqslant (D, e_D)$ and $(R, e_R) \leqslant (D, e_D)$. If x is an element of G such that $(Q, e_Q)^x \leqslant (R, e_R)$, then the morphism $\phi : Q \to R$ induced by conjugation by x is included in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$.

Although we have denoted it like a fusion system, we need to know that $\mathcal{F}_{(D,e_D)}(G, b)$ actually is a fusion system.

Theorem 2.14 Let G be a finite group and let b be a block idempotent of kG, and let (D, e_D) denote a maximal b-Brauer pair. Then $\mathcal{F}_{(D,e_D)}(G, b)$ is a fusion system on D.

The defect group and fusion system of a block yield strong information about the properties of a block.

Theorem 2.15 Suppose that G is a finite group and let b be a block idempotent with trivial defect group. Then

(i) There is a single kG-module S belonging to b, and it is projective and simple.

(ii) The ideal kGb is isomorphic with a matrix algebra $M_{\dim S}(k)$.

Furthermore, if b is any block idempotent of kG such that kGb is a matrix algebra, then b has trivial defect group.

Proof: Suppose that b is a block idempotent with trivial defect group. Then $b = \mathrm{Tr}_1^G x$ for some $x \in kG$. Let M and N be kGb-modules, and suppose that $\phi : M \to N$ is a surjective homomorphism. We will construct a splitting for ϕ, proving that every kGb-module is projective. Once we have done that, the fact that every module is projective implies that kGb is semi-simple. Since it is also indecomposable, it must be a simple algebra, and so a matrix algebra over k. The rest of the assertions now follow readily from known facts about matrix algebras.

Let $\theta : M \to N$ be a splitting of ϕ as vector spaces. For each $y \in N$, define

$$y\bar{\theta} = \sum_{g \in G} (yg^{-1}x)\theta g.$$

Then $\bar{\theta}$ is a kG-module splitting, as required.

The converse is omitted. □

Recall that by the principal block we mean the block to which the trivial module belongs. This is, in some sense, at the opposite end of the spectrum to blocks with trivial defect groups. This sense is given in the next theorem.

Theorem 2.16 Let G be a finite group. Then the principal block idempotent b_0 has defect groups the Sylow p-subgroups S of G, and

$$\mathcal{F}_{(S,e_S)}(G, b_0) = \mathcal{F}_S(G).$$

To end this chapter, we will make several observations about block fusion systems.

(i) A block b is called *nilpotent* if $\mathcal{F}_{(D,e_D)}(G, b) = \mathcal{F}_D(D)$. Nilpotent blocks have very nice properties: for example, they have a single simple module, which is endo-permutation (i.e., $M \otimes M^*$ is a permutation module).

(ii) It is not known whether there exists a block fusion system that is not a fusion system of some finite group. As mentioned in the previous chapter, there is no finite group G with a Sylow 2-subgroup S such that S is of $\mathrm{Spin}_7(3)$-type, $\mathcal{F}_S(\mathrm{Spin}_7(3)) \subseteq \mathcal{F}_S(G)$, and all involutions are G-conjugate. Kessar [15] has proved, using the classification of the finite simple groups, that no 2-block of any finite group can have such a fusion system.

(iii) Using the language of fusion systems, it is possible to state Alperin's weight conjecture – one of the most important conjectures in modular representation theory – in a way that involves the block fusion system. It is hoped that advances in our understanding of fusion systems might help with understanding these deep conjectures.

Chapter 3

Basics of Fusion Systems

This chapter develops the basic theory of fusion systems, starting with the definition and the concept of a saturated fusion system – which in some sense resembles the fusion pattern of a finite group – and dealing with normalizer and centralizer fusion systems, centric and radical subgroups, and ending with a treatment of a strengthened Alperin's fusion theorem.

3.1 The Equivalent Definitions

Here we will develop the theory of fusion systems from scratch. We begin by recalling the definition of a fusion system.

Definition 3.1 Let P be a finite p-group. Then a *fusion system* $\mathcal{F}$ on P is a category, whose objects are all subgroups of P, and whose morphisms $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ are subsets of all injective homomorphisms $Q \to R$, where Q and R are subgroups of P, with composition of morphisms given by the usual composition of homomorphisms. The sets $\mathrm{Hom}_{\mathcal{F}}(Q,R)$ should satisfy the following three axioms:

(i) for each $g \in P$ with $Q^g \leqslant R$, the associated conjugation map $\theta_g : Q \to R$ is in $\mathrm{Hom}_{\mathcal{F}}(Q,R)$;

(ii) for each $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$, the isomorphism $Q \to Q\phi$ lies in $\mathrm{Hom}_{\mathcal{F}}(Q,Q\phi)$; and

(iii) if $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q,R)$ is an isomorphism, then its inverse $\phi^{-1} : R \to Q$ lies in $\mathrm{Hom}_{\mathcal{F}}(R,Q)$.

Definition 3.2 Let P be a finite p-group, and let Q be a subgroup of P. Let $\mathcal{F}$ be a fusion system on P.

(i) The subgroup Q is said to be *fully centralized* if, whenever $\phi : Q \to R$ is an isomorphism in $\mathcal{F}$, we have that

$$|\,\mathrm{C}_P(Q)| \geqslant |\,\mathrm{C}_P(R)|.$$

(ii) The subgroup Q is said to be *fully normalized* if, whenever $\phi : Q \to R$ is an isomorphism in $\mathcal{F}$, we have that

$$|\,\mathrm{N}_P(Q)| \geqslant |\,\mathrm{N}_P(R)|.$$

Write $\mathcal{F}^{\mathrm{f}}$ for the set of all fully normalized subgroups of P.

Definition 3.3 Let P be a finite p-group, and let $\mathcal{F}$ be a fusion system on P. We say that $\mathcal{F}$ is *saturated* if

(i) $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$, and

(ii) every morphism $\phi : Q \to P$ in $\mathcal{F}$ such that $Q\phi$ is fully normalized extends to a morphism $\bar{\phi} : N_\phi \to P$, where

$$N_\phi = \{x \in \mathrm{N}_P(Q) \,:\, \text{there exists } y \in \mathrm{N}_P(Q\phi) \text{ such that } (g^x)\phi = (g\phi)^y \text{ for all } g \in Q\}.$$

The second axiom in this definition is called the *extension axiom*.

This is not the definition of a saturated fusion system given by Broto, Levi, and Oliver, for example. They prefer the following definition, which we will called 'strongly saturated' for now.

Definition 3.4 Let $\mathcal{F}$ be a fusion system on a finite p-group P. Then $\mathcal{F}$ is called *strongly saturated* if

(i) every fully normalized subgroup Q is fully centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$, and

(ii) every morphism $\phi : Q \to P$ in $\mathcal{F}$ such that $Q\phi$ is fully centralized extends to a morphism $\bar{\phi} : N_\phi \to P$, where

$$N_\phi = \{x \in \mathrm{N}_P(Q) \,:\, \text{there exists } y \in \mathrm{N}_P(Q\phi) \text{ such that } (g^x)\phi = (g\phi)^y \text{ for all } g \in Q\}.$$

Clearly every strongly saturated fusion system is saturated. We will show that the two definitions are equivalent.

Proposition 3.5 (Stancu) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let Q be a subgroup of P. Then the following are equivalent:

(i) Q is fully normalized; and

(ii) Q is fully centralized and $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.

Proof: Firstly, suppose that Q is fully normalized, and let R be a subgroup of P that is $\mathcal{F}$-isomorphic with Q such that R is fully centralized. Let $\phi : R \to Q$ be an isomorphism in $\mathcal{F}$; by the extension axiom the map ϕ extends to an injective map $\bar{\phi} : R\,\mathrm{C}_G(R) \to P$. The image of ϕ must be contained within $Q\,\mathrm{C}_G(Q)$, and so Q is fully centralized, as claimed.

Now suppose that Q is fully normalized, but that $\mathrm{Aut}_P(Q)$ is not a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Choose Q to be of maximal order with this property; certainly Q is not equal to P. Choose an automorphism ϕ of p-power order in $\mathrm{Aut}_{\mathcal{F}}(Q) \setminus \mathrm{Aut}_P(Q)$ such that $\langle\phi\rangle$ normalizes $\mathrm{Aut}_P(Q)$, which exists since $\mathrm{Aut}_P(Q)$ is not a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.

Since ϕ normalizes $\mathrm{Aut}_P(Q)$, for every $x \in \mathrm{N}_P(Q)$, there is some $y \in \mathrm{N}_P(Q)$ such that $(g^x)\phi = (g\phi)^y$ for all $g \in Q$. Therefore $N_\phi = \mathrm{N}_P(Q)$ and, since Q is fully normalized, there is an extension $\bar{\phi}$ of ϕ to the whole of $\mathrm{N}_P(Q)$. Since ϕ has p-power order, we may assume that $\bar{\phi}$ has p-power order (by raising $\bar{\phi}$ to a suitable power).

Let ψ be a map in $\mathcal{F}$ from $\mathrm{N}_P(Q)$ such that its image, R, is fully normalized. We see that $(\bar{\phi})^\psi$ is a p-element of $\mathrm{Aut}_{\mathcal{F}}(R)$. By maximal choice of Q, we see that $\mathrm{Aut}_P(R)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$, and so $(\bar{\phi})^\psi$ may be conjugated into $\mathrm{Aut}_P(R)$; hence we may choose ψ so that $(\bar{\phi})^\psi \in \mathrm{Aut}_P(R)$. Thus there is some $g \in \mathrm{N}_P(R)$ such that $x(\bar{\phi})^\psi = x^g$ for all $x \in R$.

Since $\bar{\phi}|_Q = \phi$, we see that $Q\psi$ is invariant under $(\bar{\phi})^\psi$, and so g normalizes $Q\psi$. However, Q is fully normalized, and so $\mathrm{N}_P(Q)\psi$ contains $\mathrm{N}_P(Q\psi)$. Therefore $g \in \operatorname{im}\psi$, and so if h denotes the preimage of g, we have that $x\bar{\phi} = x^h$ for all $x \in Q$.

Now we see a contradiction: in fact, ϕ may be defined by conjugation, and so lives in $\mathrm{Aut}_P(Q)$, whereas it was chosen not to. Hence $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$, as required.

The converse is much easier: it is clear that $|\,\mathrm{N}_P(Q)| = |\,\mathrm{C}_P(Q)| \cdot |\,\mathrm{Aut}_P(Q)|$. Now let Q be a fully centralized subgroup with $\mathrm{Aut}_P(Q)$ a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$, and let R be a fully normalized subgroup $\mathcal{F}$-isomorphic to Q. Then $|\,\mathrm{C}_P(Q)| = |\,\mathrm{C}_P(R)|$, and $|\,\mathrm{Aut}_{\mathcal{F}}(Q)| = |\,\mathrm{Aut}_{\mathcal{F}}(R)|$, so Q is fully normalized, as claimed. $\square$

This proves that (i) in the definition of a strongly saturated fusion system is a property that is always satisfied by a saturated fusion system. To prove that (ii) is always satisfied, we will firstly prove a proposition of independent interest.

Proposition 3.6 Let $\mathcal{F}$ be a fusion system on a finite p-group P. Let Q and R be $\mathcal{F}$-isomorphic subgroups of P such that R is fully normalized. Then there is some isomorphism $\phi : Q \to R$ such that $N_\phi = \mathrm{N}_P(Q)$.

Proof: Let ϕ be any isomorphism $\phi : Q \to R$. The group $\mathrm{Aut}_P(Q)^\phi$ is a p-subgroup of

$\mathrm{Aut}_{\mathcal{F}}(R)$, and since R is fully normalized, $\mathrm{Aut}_P(R)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(R)$, so there is some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\mathrm{Aut}_P(Q)^{\phi\alpha}$ is contained within $\mathrm{Aut}_P(R)$; set $\psi = \phi\alpha$. Then this statement means that for any x in $\mathrm{N}_P(Q)$, there is $y \in \mathrm{N}_P(R)$ such that $\psi^{-1}\theta_x\psi = \theta_y$; i.e., that $\theta_x\psi = \psi\theta_y$ for all $g \in Q$. This implies that $N\psi = \mathrm{N}_P(Q)$, as required. □

We now complete the proof of the equivalence of the two definitions.

Proposition 3.7 (Stancu) Every saturated fusion system is strongly saturated.

Proof: Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a subgroup of P. Suppose that $\phi : Q \to R$ is an isomorphism where R is fully centralized. Let $\psi : R \to P$ be a map such that $R\psi$ is fully normalized. By Proposition 3.6, we may choose ψ so that $N_\psi = \mathrm{N}_P(R)$, and in particular there is a map $\bar{\psi} : \mathrm{N}_P(R) \to P$ extending R. Then N_ϕ must be contained within $N_{\phi\psi}$, since $N_\psi = \mathrm{N}_P(R)$ and so if $\bar{\phi}$ extends ϕ to a subgroup of $\mathrm{N}_P(Q)$ then $\bar{\phi}\bar{\psi}$ also extends $\phi\psi$ to the same subgroup of $\mathrm{N}_P(Q)$. Therefore $\phi\psi$ extends to some morphism $\theta : N_\phi \to P$.

The final point is to notice that θ, composed with the inverse of $\bar{\psi}$ restricted to $N_{\phi\theta} \leqslant \mathrm{im}\,\sigma$ is a map extending ϕ to all of N_ϕ. □

From now on therefore we will abandon the notion of a strongly saturated fusion system, and feel free to use either definition as and when.

3.2 Local Subsystems

The local subsystems are the centralizer and normalizer subsystems of a given subgroup. We will define them now.

Definition 3.8 Let $\mathcal{F}$ be a fusion system on the finite p-group P. Let Q be a subgroup of P.

(i) The fusion system $\mathrm{C}_{\mathcal{F}}(Q)$ is the category whose objects are all subgroups of $\mathrm{C}_P(Q)$, and whose morphisms $\mathrm{Hom}_{\mathrm{C}_{\mathcal{F}}(Q)}(R, S)$ are

$$\{\phi \in \mathrm{Hom}_{\mathcal{F}}(R, S) \;:\; \phi \text{ extends to } \bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(QR, QS) \text{ with } \bar{\phi}|_Q = 1\}.$$

The fusion system $\mathrm{C}_{\mathcal{F}}(Q)$ is called the *centralizer* in $\mathcal{F}$ of Q.

(ii) The fusion system $\mathrm{N}_{\mathcal{F}}(Q)$ is the category whose objects are all subgroups of $\mathrm{N}_P(Q)$, and whose morphisms $\mathrm{Hom}_{\mathrm{N}_{\mathcal{F}}(Q)}(R, S)$ are

$$\{\phi \in \mathrm{Hom}_{\mathcal{F}}(R, S) \;:\; \phi \text{ extends to } \bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(QR, QS) \text{ with } \bar{\phi}|_Q \in \mathrm{Aut}_{\mathcal{F}}(Q)\}.$$

The fusion system $\mathrm{N}_{\mathcal{F}}(Q)$ is called the *normalizer* in $\mathcal{F}$ of Q.

We need to check that $\mathrm{C}_{\mathcal{F}}(Q)$ and $\mathrm{N}_{\mathcal{F}}(Q)$ are actually fusion systems.

Theorem 3.9 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a subgroup of P.

(i) The categories $\mathrm{C}_{\mathcal{F}}(Q)$ and $\mathrm{N}_{\mathcal{F}}(Q)$ are fusion systems on $\mathrm{C}_P(Q)$ and $\mathrm{N}_P(Q)$ respectively.

(ii) If $\mathcal{F}$ is saturated, then $\mathrm{C}_{\mathcal{F}}(Q)$ is saturated whenever Q is fully centralized, and $\mathrm{N}_{\mathcal{F}}(Q)$ is saturated whenever Q is fully normalized.

Proof: To prove (i), we will check the axioms for a fusion system, noting that the objects in the respective categories are correct. Thus let R and S be subgroups of $\mathrm{C}_P(Q)$. For $g \in \mathrm{C}_P(Q)$, if $\theta_g : R \to S$ is a conjugation map it clearly extends to a map $QR \to QS$ that acts trivially on Q. Thus the first axiom of a fusion system is satisfied by both $\mathrm{C}_{\mathcal{F}}(Q)$.

If $\phi : R \to S$ is a map in $\mathrm{C}_{\mathcal{F}}(Q)$, then it extends to a map $\bar{\phi} : QR \to QS$ that acts trivially on Q, and so clearly the isomorphism map $R \to R\phi$ also has this condition; thus $\mathrm{C}_{\mathcal{F}}(Q)$ satisfies the second axiom of a fusion system.

Suppose that $\phi : R \to S$ is an isomorphism in $\mathrm{C}_{\mathcal{F}}(Q)$. Then it extends to a map $\bar{\phi} : QR \to QS$ that acts trivially on Q. Thus $Q \cap R = Q \cap S$, and so in particular $\bar{\phi} \in \mathcal{F}$ is an isomorphism. Thus its inverse lies in $\mathcal{F}$, and so the map ϕ^{-1} has an extension $\bar{\phi}^{-1} : QS \to QR$ with the necessary properties.

The proofs for $\mathrm{N}_{\mathcal{F}}(Q)$ are almost exactly the same, and we will leave them as an exercise for the reader; this proves the first part of the theorem.

The proof of the second half of this theorem is beyond the scope of this course. (See [9, Proposition A.6].) □

In the first chapter, control of fusion was introduced as an interesting concept for finite groups. For fusion systems the notion also exists.

Definition 3.10 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a subgroup of P. Then Q is said to *control fusion* in $\mathcal{F}$ if $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$.

In Chapter 1 we proved a famous result of Burnside, that if G is a finite group with an abelian Sylow p-subgroup P, then $\mathrm{N}_G(P)$ controls G-fusion in P. A similar result holds for saturated fusion systems.

Proposition 3.11 Let $\mathcal{F}$ be a saturated fusion system on an abelian p-group P. Then $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(P)$.

Proof: Let Q be a subgroup of P; since P is abelian, Q is fully normalized. Let $\phi : Q \to P$ be a morphism in $\mathcal{F}$; since Q is fully normalized, ϕ extends to a map $\bar{\phi} : Q\,\mathrm{C}_P(Q) \to P$, and thus $\bar{\phi} \in \mathrm{Aut}_{\mathcal{F}}(P)$. Hence ϕ is a morphism in $\mathrm{N}_{\mathcal{F}}(P)$, as required. □

Frobenius' normal p-complement theorem also exists in some sense.

Theorem 3.12 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Then the following are equivalent:

(i) $\mathcal{F} = \mathcal{F}_P(P)$;

(ii) for any $Q \leqslant P$, the group $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a p-group; and

(iii) for any non-trivial, fully normalized subgroup Q of P, we have $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}_{\mathrm{N}_P(Q)}(\mathrm{N}_P(Q))$.

It is also possible to generalize the Glauberman–Thompson p-nilpotence theorem described in the first chapter.

Theorem 3.13 Let p be an odd prime, and let $\mathcal{F}$ be a saturated fusion system on P. Then $\mathcal{F} = \mathcal{F}_P(P)$ if and only if

$$\mathrm{N}_{\mathcal{F}}(\mathrm{Z}(J(P))) = \mathcal{F}_P(P).$$

3.3 Centric and Radical Subgroups

Here we will define the important notions of centric and radical subgroups. Centric subgroups are easy to define.

Definition 3.14 Let $\mathcal{F}$ be a fusion system on the finite p-group P. Then a subgroup Q is called *$\mathcal{F}$-centric* if, whenever R is $\mathcal{F}$-isomorphic to Q, then $\mathrm{C}_P(R) = \mathrm{Z}(R)$ (or equivalently $\mathrm{C}_P(R) \leqslant R$). Write $\mathcal{F}^{\mathrm{c}}$ for the set of all $\mathcal{F}$-centric subgroups.

Note that being $\mathcal{F}$-centric is a property invariant under $\mathcal{F}$-isomorphism, and that any $\mathcal{F}$-centric subgroup is fully centralized (since $|\,\mathrm{C}_P(R)|$ is the same order, namely $|\mathrm{Z}(R)|$, for any subgroup R that is $\mathcal{F}$-isomorphic to Q). While there is not a converse, there is a 'partial' converse in some sense.

Lemma 3.15 Let P be a finite p-group, and let $\mathcal{F}$ be a saturated fusion system. Let Q be a fully centralized subgroup of P. Then $Q\,\mathrm{C}_P(Q)$ is $\mathcal{F}$-centric.

Proof: Let $\phi : Q\,\mathrm{C}_P(Q) \to R$ be an isomorphism, and let $\theta : Q\phi \to Q$ be the inverse of $\phi|_Q$. We know that Q is fully centralized and so θ extends to a map $\bar{\theta}$ on $(Q\phi)(\mathrm{C}_P(Q\phi))$, and the image of $\bar{\theta}$ must be contained within $Q\,\mathrm{C}_P(Q)$. Now $Q\phi \leqslant R$ and so $R\,\mathrm{C}_P(R) \leqslant Q\phi\,\mathrm{C}_P(Q\phi)$. Thus

$$|R\,\mathrm{C}_P(R)| \leqslant |(Q\phi)(\mathrm{C}_P(Q\phi))| = |R|,$$

which proves that $\mathrm{C}_P(R) \leqslant R$, as required. $\square$

The set of all $\mathcal{F}$-centric subgroups is also closed under inclusion.

Lemma 3.16 Let $\mathcal{F}$ be a fusion system on P, and let Q and R be subgroups of P with $Q \leqslant R$. If Q is $\mathcal{F}$-centric then so is R, and $\mathrm{Z}(Q) \leqslant \mathrm{Z}(R)$.

Proof: Let $\phi : R \to S$ be an isomorphism in $\mathcal{F}$; since Q is $\mathcal{F}$-centric, then $\mathrm{C}_P(R\phi) \leqslant \mathrm{C}_P(Q\phi) \leqslant Q\phi \leqslant R\phi$, and so R is $\mathcal{F}$-centric. Also, $\mathrm{Z}(R\phi) = \mathrm{C}_P(R\phi) \leqslant \mathrm{C}_P(Q\phi) = \mathrm{Z}(Q\phi)$, and letting $\phi = 1$ gives us the second statement. $\square$

Let Q be a subgroup of a finite p-group P, and suppose that $\mathcal{F}$ is a fusion system on P. Notice that $\mathrm{Aut}_Q(Q)$ is a p-group, and in fact $\mathrm{Aut}_Q(Q) = \mathrm{Inn}(Q)$ is a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.

Definition 3.17 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a subgroup of P. We say that Q is *radical* if

$$\mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(Q)) = \mathrm{Inn}(Q).$$

Write $\mathcal{F}^{\mathrm{r}}$ for the set of all radical subgroups.

We will extend our notation in an obvious fashion, and refer to, for example, $\mathcal{F}^{\mathrm{frc}}$ for the set of all fully normalized, $\mathcal{F}$-centric, radical subgroups of $\mathcal{F}$.

Recall from Chapter 1 the following definition. Let G be a finite group with $p \mid |G|$, and let M be a subgroup of G. We say that M is *strongly p-embedded* if M contains a Sylow p-subgroup of G, and $M \cap M^g$ is a p'-group for all $g \in G \setminus M$. Note that if G has a strongly p-embedded subgroup then $\mathrm{O}_p(G) = 1$.

Definition 3.18 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a subgroup of P. We say that Q is *$\mathcal{F}$-essential* if Q is $\mathcal{F}$-centric, and $\mathrm{Out}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}}(Q)/\,\mathrm{Inn}(Q)$ contains a strongly p-embedded subgroup.

Notice that every essential subgroup is radical.

Proposition 3.19 Let $\mathcal{F}$ be a fusion system on a finite p-group P. Let Q be a subgroup such that $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$. Then Q is contained within every centric, radical subgroup of P.

Proof: Let R be a subgroup of P, and suppose that R is radical and centric. We claim that the image of Q in $\mathrm{Aut}_{\mathcal{F}}(R)$ is, in fact, a normal subgroup. If this is true, then it is contained in the image $\mathrm{Inn}(R) = \mathrm{O}_p(\mathrm{Aut}_{\mathcal{F}}(R))$ of R in $\mathrm{Aut}_{\mathcal{F}}(R)$. Thus $Q \leqslant R\,\mathrm{C}_P(R)$, and since R is centric, $R\,\mathrm{C}_P(R) = R$, yielding the result.

It remains to prove the claim. Let ϕ be an automorphism in $\mathrm{Aut}_{\mathcal{F}}(R)$, and extend ϕ to an automorphism of QR. Note that both Q and R are ϕ-invariant, and so $\mathrm{N}_Q(R) = Q \cap \mathrm{N}_{QR}(R)$ is ϕ-invariant.

If $g \in Q$, then g normalizes R if and only if $\theta_g \in \mathrm{Aut}_Q(R)$, and so it suffices to show that $\phi^{-1}\theta_g\phi = \theta_{g\phi}$, for $\theta_g \in \mathrm{Aut}_Q(R)$. This calculation is well-known:

$$x(\phi^{-1}\theta_g\phi) = (x\phi^{-1})\theta_g\phi = (g^{-1}x\phi^{-1}g)\phi = (g\phi)^{-1}x(g\phi),$$

as claimed. □

Groups with a strongly p-embedded subgroup can be characterized in terms of their Quillen complex. Rather than deal with the whole Quillen complex, we simply consider the partially ordered set of all non-identity p-subgroups of a finite group, which we will turn into an undirected graph in the obvious way, and denote this by $A_p(G)$.

Proposition 3.20 Let G be a finite group such that $p \mid |G|$. Then G has a strongly p-embedded p-subgroup if and only if the graph $A_p(G)$ is disconnected.

Proof: Suppose that G has a strongly p-embedded subgroup, M, containing a Sylow p-subgroup P. Let g be an element of $G \setminus M$, and consider P^g. We claim that P^g and P lie in different components of $A_p(G)$. Since $M \cap M^g$ is a p'-group, we see that $P \cap P^g = 1$. Suppose that $Q = Q_0, Q_1, \dots, Q_n = Q^g$ is a path of minimal length linking $Q \leqslant P$ and $Q^g \leqslant P^g$, as we range over all subgroups of P and all paths. Since $Q \leqslant P$ and $Q \cap Q_1 \neq 1$, we must have that Q_1 is contained within P, contradicting the minimal length claim. Thus P and P^g lie in different components, as claimed.

Now suppose that $A_p(G)$ is disconnected, and let P be a Sylow p-subgroup. Since $A_p(G)$ is disconnected, this splits $\mathrm{Syl}_p(G)$ into (at least two) components (else all p-subgroups, which are contained in Sylow p-subgroups, would be connected to each other), and let $\mathcal{S}$ denote the subset of $\mathrm{Syl}_p(G)$ lying in the same component as P. Let M denote the set of all $g \in G$ such that $P^g \in \mathcal{S}$. The claim is that M is a strongly p-embedded subgroup. Firstly, M is clearly a subgroup, and contains a Sylow p-subgroup. Furthermore, if $g \notin M$, then for any (non-trivial) p-subgroup Q of M, we have that Q and Q^g are not connected in $A_p(G)$, so certainly $Q \cap Q^g = 1$. Hence $M \cap M^g$ is a p'-group, as required. □

3.4 Alperin's Fusion Theorem

In this section we will provide a proof of Alperin's fusion theorem for fusion systems. In its original statement, it essentially ran as follows: any $\mathcal{F}$-isomorphism may be 'factored' into restrictions of automorphisms of fully normalized, centric, radical subgroups of the ambient p-group P. In the refined version that we give here, the class of subgroups needed to factor an automorphism is restricted still further, with the loss of granularity being an automorphism of P itself.

Theorem 3.21 (Alperin's fusion theorem) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, let $\mathscr{S}$ denote the set of all fully normalized, essential subgroups of P, and let Q and R denote two subgroups of P, with $\phi : Q \to R$ an $\mathcal{F}$-isomorphism. Then there exists

(i) a sequence of $\mathcal{F}$-isomorphic subgroups $Q = Q_0$, Q_1, $\dots$, $Q_{n+1} = R$,

(ii) a sequence S_1, S_2, $\dots$, S_n of elements of $\mathscr{S}$, with $Q_{i-1},\ Q_i \leqslant S_i$,

(iii) a sequence of $\mathcal{F}$-automorphisms ϕ_i of S_i such that $Q_{i-1}\phi_i = Q_i$, and

(iv) an $\mathcal{F}$-automorphism ψ of P (mapping Q_n to Q_{n+1}),

such that

$$(\phi_1\phi_2\dots\phi_n\psi)|_Q = \phi.$$

Proof: We begin by showing that if θ is a $\mathcal{F}$-automorphism of P, and ρ is an $\mathcal{F}$-automorphism of some fully normalized, essential subgroup E, then there exists an $\mathcal{F}$-automorphism ρ' of some other fully normalized, essential subgroup E' such that $\theta\rho = \rho'\theta$. Notice that

$$u\theta\rho = u(\theta\rho\theta^{-1})\theta$$

for all $u \in E\theta^{-1}$. We need to show that $E' = E\theta^{-1}$ is a fully normalized, essential subgroup, for then $\rho' = \theta\rho\theta^{-1}$ is an automorphism of it, and we have proved our claim. However, $\theta \in \mathrm{Aut}_{\mathcal{F}}(P)$, and so $\mathrm{N}_P(E)\theta^{-1} = \mathrm{N}_P(E\theta^{-1})$. Since E is fully normalized, $|\,\mathrm{N}_P(E)|$ is maximal amongst subgroups $\mathcal{F}$-isomorphic to E, and so therefore $E\theta^{-1}$ is fully normalized as well. The property of being essential is clearly transported by θ^{-1} and so the claim holds.

This proves that the product of two $\mathcal{F}$-isomorphisms that possess a decomposition of the required form also possesses a decomposition of the required form, as does the inverse of such an $\mathcal{F}$-isomorphism. This will be invaluable in what follows.

We proceed by reverse induction on $|Q|$, a subgroup of P. If $Q = P$ then ϕ is an automorphism of P, and so $n = 0$ and the theorem is true. Thus we may assume that $Q < P$. The proof will proceed in stages.

Suppose firstly that R is fully normalized. By Proposition 3.6, there is a map ϕ' from Q to R such that $N_{\phi'} = \mathrm{N}_P(Q)$. Since any two isomorphisms between two subgroups differ by an automorphism of R, there exists $\chi \in \mathrm{Aut}_{\mathcal{F}}(R)$ such that $\phi\chi = \phi'$. Thus there is a morphism $\overline{\phi\chi} : \mathrm{N}_P(Q) \to P$ extending $\phi\chi$, and since $Q < \mathrm{N}_P(Q)$, we may apply reverse induction to $\overline{\phi\chi}$, to get that this morphism, and hence $\phi\chi$, has such a decomposition.

It remains to show that χ has such a decomposition, since then $\phi = (\phi\chi)\chi^{-1}$ has a decomposition in the required form. Thus let χ be an element of $\mathrm{Aut}_{\mathcal{F}}(R)$, where R is fully normalized. If R is not centric, then $R\,\mathrm{C}_P(R) > R$, and since N_χ contains $R\,\mathrm{C}_P(R)$, we may decompose $\bar{\chi}$ (which extends χ to $R\,\mathrm{C}_P(R)$), so we may decompose χ, as claimed.

Since R is fully normalized, it cannot be essential, since else χ would be of the required form.

By Proposition 3.20, there exists two sequences of subgroups $\mathrm{Aut}_P(R) = A_1, A_2, \dots, A_n = \mathrm{Aut}_P(R)^\chi$ and $B_1, \dots, B_{n-1}$ such that

(i) $B_i, \leqslant A_i, A_{i+1}$ for $i < n$, and

(ii) $\mathrm{Aut}_R(R) < B_i$ for all i.

Replacing the A_i with Sylow p-subgroups of $\mathrm{Aut}_{\mathcal{F}}(R)$, we may suppose that there are θ_i such that $A_i^{\theta_i} = A_{i+1}$. Write $\chi_i = \theta_0\theta_1 \dots \theta_i$ and $\chi = \chi_n$.

Recall that the subgroup N_ϕ is the largest subgroup of $\mathrm{N}_P(Q)$ such that $(N_\phi / \mathrm{C}_P(Q))^\phi \leqslant \mathrm{Aut}_P(Q\phi)$. We see that $N_{\theta_i}/\mathrm{Z}(Q)$ contains $B_i^{\chi_i^{-1}}$, since

$$\left(B_i^{\chi_i^{-1}}\right)^{\theta_i} = B_i^{\chi_{i-1}^{-1}} \leqslant A_i^{\chi_{i-1}^{-1}} = A_1.$$

Thus N_{θ_i} strictly contains Q. Since Q is fully normalized, θ_i extends to a map from N_{θ_i}, and this map has a decomposition of the required form, whence θ_i does. Finally, the composition of the θ_i is χ, and so that has a decomposition of the required form.

The last step is to remove the assumption that R is fully normalized. Let $\nu : Q \to S$ be an $\mathcal{F}$-isomorphism such that S is fully normalized. Now both ν and $\phi^{-1}\nu$ have decompositions of the required form, since they are $\mathcal{F}$-isomorphisms mapping onto a fully normalized subgroup. Therefore ϕ has such a decomposition, by the conclusion of the first paragraph. □

A weaker form of Alperin's fusion theorem is also useful, and in most cases is all that is needed for applications.

Theorem 3.22 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\phi : Q \to R$ be an isomorphism. Then there exists

(i) a sequence of $\mathcal{F}$-isomorphic subgroups $Q = Q_0, Q_1, \dots, Q_{n+1} = R$,

(ii) a sequence $S_1, S_2, \dots, S_n$ of fully normalized, $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroups, with $Q_{i-1}, Q_i \leqslant S_i$, and

(iii) a sequence of $\mathcal{F}$-automorphisms ϕ_i of S_i such that $Q_{i-1}\phi_i = Q_i$,

such that

$$(\phi_1\phi_2 \dots \phi_n)|_Q = \phi.$$

Proof: Since every essential subgroup is radical and centric, we have expanded the collection of subgroups for which we may consider automorphisms. In particular, the whole group P is fully normalized, centric, and radical (since $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$), and so an $\mathcal{F}$-automorphism of P, as required by Theorem 3.21, is allowed as one of the ϕ_i. Thus Theorem 3.21 implies this weaker version, as claimed. □

The question of whether a fusion system $\mathcal{F}$ on a finite p-group P has any fully normalized, essential subgroups is an interesting one. One may turn the question on its head, and ask whether a particular p-group may be an essential subgroup of some overgroup. Indeed, can the automorphism group of a p-group contain a strongly p-embedded subgroup at all?

It is known (Martin, Henn–Priddy) that for almost all p-groups, the automorphism group (and hence the outer automorphism group) is itself a p-group. Hence almost all p-groups cannot be found as essential subgroups.

Proposition 3.23 Let P be a dihedral 2-group of order at least 8, a quaternion 2-group of order at least 16, or a semidihedral 2-group of order at least 32. Then $\mathrm{Aut}(P)$ is a p-group.

The next theorem deals with metacyclic p-groups, where p is odd.

Theorem 3.24 (Stancu) Let P be a metacyclic p-group, where p is odd.

(i) If P is isomorphic with $C_n \times C_m$, where $n \neq m$, then P cannot be an essential subgroup in any saturated fusion system.

(ii) If P is non-abelian then P is not an essential subgroup in any saturated fusion system.

We omit a (largely unenlightening) proof of this fact, since it is not relevant to our discussion.

To end this chapter, we include a brief discussion of so-called resistant p-groups.

Definition 3.25 A p-group is called *resistant* if, for any saturated fusion system $\mathcal{F}$, it is true that P controls fusion in $\mathcal{F}$; i.e., if whenever $\mathcal{F}$ is a saturated fusion system on P, then we have $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(P)$.

Lemma 3.11 states that abelian p-groups are resistant. Let $\mathcal{F}$ be a fusion system on a finite p-group P. Clearly if P contains no fully normalized, essential subgroups then $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(P)$, and this condition is also necessary.

Stancu has also shown that metacyclic p-groups (for p odd) are resistant, and has developed an elegant equivalent condition for control of fusion, using weakly and strongly closed subgroups. (We will meet these in the next chapter.)

Chapter 4

Normal Subsystems, Quotients, and Morphisms

A morphism of a fusion system is a natural notion, and quite easy to define. However, there is some debate in the subject currently about the correct definition of a 'normal' subsystem, one due to Aschbacher, and the other due to Linckelmann–Puig. Part of the problem is to do with precisely why you want the notion in the first place: for example, Aschbacher would like to use arguments from local finite group theory, and so his definition of a normal subsystem is tailored for that use. Here we will use Linckelmann's definition, which is strictly weaker than the Aschbacher definition; we will call Aschbacher's version of normal a 'strongly normal' subsystem. To recapitulate, Linckelmann–Puig normality is denoted here by 'normal', and Aschbacher normality is simply called 'strong normality'.

4.1 Morphisms of Fusion Systems

Since a fusion system is a category on all subgroups of a p-group, it makes sense to make the following definition.

Definition 4.1 Let $\mathcal{F}$ and $\mathcal{E}$ be fusion systems on the finite p-groups P and Q respectively. Then a *morphism* $\phi : \mathcal{F} \to \mathcal{E}$ of fusion systems is a pair $(\phi, \{\phi_{R,S} : R, S \leqslant P\})$, where $\phi : P \to Q$ is a group homomorphism, and for each $R, S \leqslant P$, the map $\phi_{R,S}$ is a function

$$\phi_{R,S} : \mathrm{Hom}_{\mathcal{F}}(R, S) \to \mathrm{Hom}_{\mathcal{E}}(R\phi, S\phi)$$

such that the corresponding map $\mathcal{F} \to \mathcal{E}$ on the category forms a functor.

Notions of kernels, injectivity and surjectivity are natural.

Definition 4.2 Let $\mathcal{F}$ and $\mathcal{E}$ by fusion systems on the finite p-groups P and Q respectively, and let $\phi : \mathcal{F} \to \mathcal{E}$ be a morphism.

(i) The *kernel* of ϕ is the kernel of the underlying group homomorphism $P \to Q$, necessarily a normal subgroup of P.

(ii) The map ϕ is said to be *injective* if $\ker\phi = 1$.

(iii) The map ϕ is said to be *surjective* if the map $P \to Q$ is surjective, and for any two subgroups R and S of Q, the map

$$\phi_{R',S'} : \mathrm{Hom}_{\mathcal{F}}(R', S') \to \mathrm{Hom}_{\mathcal{E}}(R, S)$$

is a surjective map, where R' and S' are the preimages of R and S under ϕ.

Let K and H be subgroups of the finite group G, with $K \leqslant H$. Recall that a K is said to be *strongly closed* in H with respect to G if, for all $x \in K$, we have that $x^G \cap H \leqslant K$; that is, all G-conjugates of elements of K that lie inside H lie inside K. The corresponding definition for fusion systems is below.

Definition 4.3 Let $\mathcal{F}$ be a fusion system on the finite p-group P, and let Q be a subgroup of P. Then Q is said to be *strongly $\mathcal{F}$-closed* if, for each $R \leqslant Q$ and $S \leqslant P$, and for each $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, S)$, we have that $R\phi \leqslant Q$.

The following verification is easy, and left to the reader.

Lemma 4.4 Let G be a finite group, and let P be a Sylow p-subgroup of G. Let Q be a subgroup of P, and write $\mathcal{F}$ for $\mathcal{F}_P(G)$. Then Q is strongly closed in P with respect to G if and only if Q is strongly $\mathcal{F}$-closed.

The reason for introducing strongly closed subgroups now is the following proposition.

Proposition 4.5 Let $\mathcal{F}$ be a fusion system on the finite p-group P. Let ϕ be a morphism from $\mathcal{F}$. Then $\ker\phi$ is strongly $\mathcal{F}$-closed.

Proof: Let Q be the kernel of ϕ, and let R be a subgroup of Q. We need to show that if S is $\mathcal{F}$-isomorphic to R then $S \leqslant Q$. Let $\psi : R \to S$ be an isomorphism. Then $\psi\phi$ is an isomorphism in the target fusion system, and since $S\phi$ is trivial, we must have that $R\phi$ is trivial also. Thus Q is strongly $\mathcal{F}$-closed. □

Thus to every surjective morphism of fusion systems, one may associate a strongly $\mathcal{F}$-closed subgroup, namely its kernel. In fact, the map ϕ is determined by the underlying group homomorphism, but this will not be proved here.

We now consider a construction of Puig. Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a strongly $\mathcal{F}$-closed subgroup. We will construct a fusion system on P/Q, which we denote by $\mathcal{F}/Q$. The objects of the category are all subgroups of P/Q, and the morphisms of $\mathcal{F}/Q$ are all morphisms $\bar{\phi} : R/Q \to S/Q$ induced from $\phi : R \to S$. (Since Q is strongly $\mathcal{F}$-closed, ϕ induces an automorphism of Q.)

Proposition 4.6 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a strongly $\mathcal{F}$-closed subgroup.

(i) The category $\mathcal{F}/Q$ is a fusion system on P/Q.

(ii) If $\mathcal{F}$ is saturated the $\mathcal{F}/Q$ is saturated.

Proof: Certainly $\mathcal{F}_{P/Q}(P/Q)$ is contained within $\mathcal{F}/Q$, since conjugation by a coset Qx on P/Q is the same as that induced by x on P/Q. If $\bar{\phi} : R/Q \to S/Q$ is a morphism in $\mathcal{F}/Q$, then it is induced by a morphism $\phi : R \to S$. As $\mathcal{F}$ is a fusion system, the corresponding $\mathcal{F}$-isomorphism $\phi : R \to R\phi$ lies in $\mathcal{F}$, and since Q is strongly $\mathcal{F}$-closed, Q lies inside both R and $R\phi$. The second axiom of a fusion system is satisfied by $\mathcal{F}/Q$ because $\bar{\phi} : R/Q \to (R/Q)\bar{\phi}$ is induced by $\phi : R \to R\phi$. Finally, if $\phi : R \to S$ and $\psi : S \to R$ are $\mathcal{F}$-isomorphisms with $Q \leqslant R, S$ and $\phi\psi = 1$, then the induced morphisms $\bar{\phi}$ and $\bar{\psi}$ are mutually inverse as well, proving that $\mathcal{F}/Q$ is, indeed, a fusion system.

We proceed with the proof of (ii). Assume that $\mathcal{F}$ is a saturated fusion system. All automorphisms in $\mathrm{Aut}_{\mathcal{F}/Q}(P/Q)$ are induced from automorphisms in $\mathrm{Aut}_{\mathcal{F}}(P)$, and so the obvious homomorphism $\mathrm{Aut}_{\mathcal{F}}(P) \to \mathrm{Aut}_{\mathcal{F}/Q}(P/Q)$ is surjective. The image of $\mathrm{Aut}_P(P)$ in $\mathrm{Aut}_{\mathcal{F}/Q}(P/Q)$ is clearly $\mathrm{Aut}_{P/Q}(P/Q)$, so that it satisfies the first axiom of a saturated fusion system.

Suppose that $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}/Q}(R/Q, S/Q)$ is an isomorphism such that S/Q is fully $\mathcal{F}/Q$-normalized. We claim that S is also fully $\mathcal{F}$-normalized. Since $Q \leqslant R$, and Q is strongly $\mathcal{F}$-closed, we have that, for all T that are $\mathcal{F}$-isomorphic to R, we have that $Q \leqslant T$ and $Q \leqslant \mathrm{N}_P(T)$. Also, $\mathrm{N}_P(R)/Q = \mathrm{N}_{P/Q}(R/Q)$. Therefore

$$|\,\mathrm{N}_P(T)| = |\,\mathrm{N}_{P/Q}(T/Q)| \cdot |Q| \leqslant |\,\mathrm{N}_{P/Q}(S/Q)| \cdot |Q| = |\,\mathrm{N}_P(S)|;$$

hence S is fully $\mathcal{F}$-normalized.

Let $\phi : R \to S$ be an isomorphism in $\mathcal{F}$ inducing $\bar{\phi}$. Our next claim is that $(N_\phi)/Q = N_{\bar{\phi}}$. If this is true, then the fact that S is fully $\mathcal{F}$-normalized means that ϕ extends to $\phi' : N_\phi \to P$, and the image of ϕ' in $\mathcal{F}/Q$ extends $\bar{\phi}$ to $N_{\bar{\phi}} = (N_\phi)/Q$, as needed.

Clearly, $N_\phi/Q \leqslant N_{\bar{\phi}}$, and so we need to prove the other inequality. Write X for the preimage of $N_{\bar{\phi}}$ in X/Q. The homomorphism $P \to P/Q$, as mentioned earlier, induces a surjection $\rho : \mathrm{Aut}_P(R) \to \mathrm{Aut}_{P/Q}(R/Q)$. Then

$$(\mathrm{Aut}_X(R)^{\phi})\rho = \mathrm{Aut}_{N_{\bar{\phi}}}(R/Q)^{\bar{\phi}} \leqslant \mathrm{Aut}_{P/Q}(S/Q).$$

Therefore $\mathrm{Aut}_X(R)^{\phi} \leqslant \mathrm{Aut}_P(S)$, and so N is contained within N_{ϕ}, as claimed. $\square$

Hence the quotients of $\mathcal{F}$ are in one-to-one correspondence with strongly $\mathcal{F}$-closed subgroups, mirroring the situation for groups, rings, and so on.

Let G be a finite group, with Q a normal p-subgroup. Let P be a Sylow p-subgroup of G. Stancu proved that this construction coincides with that of $\mathcal{F}_P(G)$ and $\mathcal{F}_{P/Q}(G/Q)$, but we will not prove this here. (See Stancu's *Quotients of fusion systems.*)

4.2 Normal Subgroups

In this section, we build upon the concept of a strongly $\mathcal{F}$-closed subgroup to produce the concept of a subgroup that is normal in a fusion system (note not a subsystem that is normal).

Definition 4.7 Let Q be a subgroup of the finite p-group P, and let $\mathcal{F}$ be a fusion system on P. Suppose that Q is strongly $\mathcal{F}$-closed. Then Q is said to be *normal* in $\mathcal{F}$ (denoted $Q \trianglelefteq \mathcal{F}$) if $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$; i.e., if, for each subgroup $R \leqslant P$ and $\phi \in \mathrm{Hom}_{\mathcal{F}}(R, P)$, the map ϕ may be extended to a map $\bar{\phi} \in \mathrm{Hom}_{\mathcal{F}}(QR, P)$ such that $\bar{\phi}|_Q$ is an automorphism of Q.

The following lemma is easy, and its proof is left to the reader.

Lemma 4.8 Let G be a finite group and let P be a Sylow p-subgroup of G. Suppose that Q is a normal p-subgroup of G. Then $Q \trianglelefteq \mathcal{F}_P(G)$.

Proposition 4.9 Let P be a finite p-group, and let $\mathcal{F}$ be a fusion system on P. Let Q be a subgroup of P. Then Q is normal in $\mathcal{F}$ if and only if Q is strongly $\mathcal{F}$-closed and Q is contained in every member of $\mathcal{F}^{\mathrm{frc}}$.

Proof: Suppose that Q is normal in $\mathcal{F}$. Then certainly Q is strongly $\mathcal{F}$-closed, and by Proposition 3.19, Q is contained within every $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup. Thus one direction of the proof is clear. (That Q is fully normalized is clear, since $P = \mathrm{N}_P(Q)$.)

Thus suppose that Q is strongly $\mathcal{F}$-closed and contained within every member of $\mathcal{F}^{\mathrm{frc}}$, and let $\phi : R \to S$ be an isomorphism in $\mathcal{F}$. We need to prove that there is some map $\bar{\phi} : QR \to P$ extending ϕ, such that $\bar{\phi}$ restricts to an automorphism of Q.

By the weak version of Alperin's fusion theorem (Theorem 3.22), ϕ may be decomposed into a sequence of isomorphisms $\phi_1 \dots \phi_n$, which are restrictions of automorphisms of T_1, T_2,

…, T_n, elements of $\mathcal{F}^{\mathrm{frc}}$. Since Q is contained in every member of $\mathcal{F}^{\mathrm{frc}}$, if we can show that every automorphism of a member of $\mathcal{F}^{\mathrm{frc}}$ restricts to an automorphism of Q, then we may extend each of the ϕ_i to a map whose domain contains Q, and hence we may extend ϕ to a map whose domain contains Q.

The fact that an $\mathcal{F}$-automorphism of a group restricts to an $\mathcal{F}$-automorphism of a strongly $\mathcal{F}$-closed subgroup is trivial: simply take the subgroup itself and the automorphism restricted to the subgroup in the definition of strong $\mathcal{F}$-closure.

Hence $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$, and so $Q \trianglelefteq \mathcal{F}$, as required. □

The next lemma tells us that strongly $\mathcal{F}$-closed and normal subgroups of fusion systems behave like normal subgroups of groups, in at least one respect.

Proposition 4.10 Let P be a finite p-group, and let Q and R be subgroups of P. Let $\mathcal{F}$ be a fusion system on P.

(i) If Q and R are strongly $\mathcal{F}$-closed then QR is strongly $\mathcal{F}$-closed.

(ii) If Q and R are normal in $\mathcal{F}$ then QR is normal in $\mathcal{F}$.

Proof: Proving (i) is beyond the scope of this course; see Aschbacher's *The generalized Fitting subsystem of a fusion system.*

To prove (ii), we use (i) and Proposition 4.9: if Q and R are normal in $\mathcal{F}$, then QR is strongly $\mathcal{F}$-closed by (i), and both Q and R are contained within every member of $\mathcal{F}^{\mathrm{frc}}$ by Proposition 4.9. Hence QR is contained within every member of $\mathcal{F}^{\mathrm{frc}}$, and so by Proposition 4.9 again, we see that $QR \trianglelefteq \mathcal{F}$. □

As a trivial consquence to this proposition, we see that there is a largest subgroup that is normal in a fusion system.

Definition 4.11 Let $\mathcal{F}$ be a fusion system on the finite p-group P. Then the largest subgroup normal in $\mathcal{F}$ is denoted by $\mathrm{O}_p(\mathcal{F})$.

4.3 Normal Fusion Systems

Let $\mathcal{F}$ be a fusion system. In this section we will define what it means for a subsystem to be normal, and consider some of its properties. The concept of an normal subsystem is *almost* what Aschbacher wants, in the sense that normal subsystems are quite close to strongly normal subsystems. We begin with the definition.

Definition 4.12 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let $\mathcal{F}'$ be a subsystem on a subgroup Q of P, where Q is strongly $\mathcal{F}$-closed. We say that $\mathcal{F}'$ is *normal* in $\mathcal{F}$ if, for each $R \leqslant S \leqslant Q$, $\phi \in \mathrm{Hom}_{\mathcal{F}'}(R, S)$, and $\psi \in \mathrm{Hom}_{\mathcal{F}}(S, P)$, we have that $\psi^{-1}\phi\psi$ is a morphism in $\mathrm{Hom}_{\mathcal{F}'}(R\psi, Q)$. We denote normality by $\mathcal{F}' \trianglelefteqslant \mathcal{F}$.

This definition is slightly different from that of Linckelmann, although it is equivalent. We will not give the alternative definition here.

The intersection of two subsystems is clear to define: if $\mathcal{E}$ and $\mathcal{E}'$ are subsystems on Q and R respectively, then $\mathcal{E} \cap \mathcal{E}'$ is the fusion system on $Q \cap R$ consisting of all morphisms of the oversystem $\mathcal{F}$ that are in both $\mathcal{E}$ and $\mathcal{E}'$.

Proposition 4.13 Let $\mathcal{F}$ be a fusion system on the finite p-group P. Let $\mathcal{E}$ and $\mathcal{E}'$ be subsystems on the subgroups Q and R respectively.

(i) If $\mathcal{E}$ is invariant in $\mathcal{F}$, then $\mathcal{E} \cap \mathcal{E}'$ is an invariant subsystem of $\mathcal{E}'$.

(ii) If both $\mathcal{E}$ and $\mathcal{E}'$ are invariant in $\mathcal{F}$, then so is $\mathcal{E} \cap \mathcal{E}'$.

Proof: Firstly assume that $\mathcal{E}$ is invariant in $\mathcal{F}$. If Q is strongly $\mathcal{F}$-closed, then it is easy to see that $Q \cap R$ is strongly $\mathcal{E}$-closed: if$\phi : S \to T$ is a morphism in $\mathcal{E}'$ with $S \leqslant Q \cap R$ then $S\phi \leqslant Q$ and $S\phi \leqslant T \leqslant R$, so $S\phi \leqslant Q \cap R$, and $Q \cap R$ is strongly $\mathcal{E}'$-closed.

Now let S and T be subgroups of $Q \cap R$, such that $S \leqslant T$. Suppose that $\phi \in \mathrm{Hom}_{\mathcal{E}\cap\mathcal{E}'}(S, T)$ and that $\psi \in \mathrm{Hom}_{\mathcal{E}'}(S, R)$. Then $\psi^{-1}\phi\psi$ is in $\mathcal{E}'$ since its components are in $\mathcal{E}'$. Also, since $\mathcal{E}$ is invariant in $\mathcal{F}$, then $\psi^{-1}\phi\psi$ is in $\mathcal{E}$, and so it is in $\mathcal{E} \cap \mathcal{E}'$. Thus $\mathcal{E} \cap \mathcal{E}'$ is invariant in $\mathcal{E}'$, proving (i).

The proof of (ii) is similar, and left to the reader. □

Proposition 4.14 Let $\mathcal{F}$ be a saturated fusion system on a p-group P. Then $\mathcal{F}_P(P)$ is normal in $\mathcal{F}$ if and only if $P \trianglelefteqslant \mathcal{F}$.

Proof: If $\mathrm{N}_{\mathcal{F}}(P) = \mathcal{F}$, then every morphism in $\mathcal{F}$ extends to an automorphism of P. Therefore, for all $g \in P$, $Q \leqslant P$, and $\psi : Q \to P$ in $\mathcal{F}$, we need to show that $\psi^{-1}\theta_g\psi : Q\psi \to P$ is a morphism in $\mathcal{F}_P(P)$. Since ψ extends to an automorphism of $\mathcal{F}$,

$$\psi^{-1}\theta_g\psi = \theta_{g\psi},$$

as claimed.

Suppose that $\mathcal{F}_P(P)$ is normal in $\mathcal{F}$. Then, for any morphism $\psi : Q \to P$ in $\mathcal{F}$ and any element $x \in \mathrm{N}_P(R)$, there is $y \in \mathrm{N}_P(R)$ such that, for all $g \in R$,

$$(g^x)\psi = (g\psi)^y.$$

If $R\psi$ is $\mathcal{F}$-centric, then $R\psi$ is fully centralized, and ψ extends to $\mathrm{N}_P(R) \to P$ in $\mathcal{F}$. By Lemma 3.16, $\mathrm{N}_P(R)$ is also $\mathcal{F}$-centric, and so by induction ψ extends to an automorphism of P. Since every morphism may be written as a combination of automorphisms of subgroups in $\mathcal{F}^{\mathrm{frc}}$ by Alperin's fusion theorem, we see that ψ extends to an automorphism of P, as claimed. $\square$

Stancu in [24] has proved that $\mathcal{F}_Q(Q)$ is normal in $\mathcal{F}$ if and only if $Q \trianglelefteqslant \mathcal{F}$, naturally extending this result for all Q.

In the case of abelian subgroups, it is very easy to understand.

Lemma 4.15 Suppose that $\mathcal{F}$ is a fusion system on P, and that Q is an abelian subgroup of P, and suppose that Q is strongly $\mathcal{F}$-closed. Then $\mathcal{F}_Q(Q)$ is normal in $\mathcal{F}$.

Proof: Since Q is abelian, the morphisms of $\mathcal{F}_Q(Q)$ are only inclusions $R \leqslant S$. If $\phi : R \to P$ is a morphism in $\mathcal{F}$, then $R\phi \leqslant Q$ (as Q is strongly $\mathcal{F}$-closed) and so $R\phi \leqslant S\phi$ is the inclusion corresponding to the inclusion $R \leqslant S$. Hence $\mathcal{F}_Q(Q)$ is normal in $\mathcal{F}$. $\square$

The main problem with normal subsystems is that they need not be saturated. This is not a problem in the Linckelmann approach, because all fusion systems are assumed to be saturated. Of course, we could repair this deficiency by considering only normal *saturated* systems. This is enough for many appropriate theorems, but there are problems with regards constraint.

Definition 4.16 Let $\mathcal{F}$ be a saturated fusion system. Then $\mathcal{F}$ is said to be *simple* if $\mathcal{F}$ contains no proper, non-trivial, normal, saturated subsystems.

The theory of simple fusion systems will be exposited in the next chapter.

4.4 Strongly Normal Fusion Systems

The theory of strongly normal subsystems was developed by Aschbacher to repair a deficiency in the theory of normal subsystems, namely that in constrained fusion systems (see later), normal subsystems should have models. (This won't make much sense at the moment, but it will.)

Definition 4.17 Let $\mathcal{F}$ be a saturated fusion system on P. Let $\mathcal{F}'$ be a subsystem of $\mathcal{F}$ on the subgroup Q. Then $\mathcal{F}$ is called *strongly normal* if $\mathcal{F}$ is saturated, normal, and each $\phi \in \mathrm{Aut}_{\mathcal{F}'}(Q)$ extends to $\bar{\phi} \in \mathrm{Aut}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))$ such that $[\bar{\phi}, \mathrm{C}_P(Q)] \leqslant \mathrm{Z}(Q)$. Write $\mathcal{F}' \preccurlyeq \mathcal{F}$ if $\mathcal{F}'$ is a strongly normal subsystem of $\mathcal{F}$.

This seems a bit of a random definition, but it seems to work, as far as the fusion systems of groups are concerned, because of the following result.

Proposition 4.18 Let $\mathcal{F} = \mathcal{F}_P(G)$ be the fusion system on a finite group G with Sylow p-subgroup P. Let H be a normal subgroup of G, and set $Q = P \cap H$; write $\mathcal{F}' = \mathcal{F}_Q(H)$. Then $\mathcal{F}' \preccurlyeq \mathcal{F}$.

We will prove this later, but for now let's see how the concept of a strongly normal subsystem might be thought of an 'unnatural'.

Example 4.19 Let G be the direct product of three copies of A_4, labelled G_1, G_2 and G_3. Let x_1 be an element of order 3 in G_i, and let $X = \langle x_1x_2, x_1x_3\rangle$. Let P_i be the Sylow 2-subgroup of G_i, and P be the Sylow 2-subgroup of G. Let $H = XS$. Then $H_1 = \langle x_1x_2, S_1, S_2\rangle$ and $H_2 = \langle x_1x_3, S_1, S_3\rangle$ are normal subgroups of H, with Sylow 2-subgroups $Q_1 = P_1P_2$ and $Q_2 = P_1P_3$. Write $\mathcal{F}_i = \mathcal{F}_{Q_i}(H_i)$. Since the H_i are normal subgroups of H, $\mathcal{F}_i \preccurlyeq \mathcal{F}$.

So far, so good. Now let $\mathcal{E} = \mathcal{F}_1 \cap F_2$. Now, $\mathcal{E}$ is a saturated, normal subsystem of $\mathcal{F}$, since both $\mathcal{F}_1$ and $\mathcal{F}_2$ are normal and saturated. However, $\mathcal{E}$ is *not* strongly normal in $\mathcal{F}$.

This example shows that not all normal, saturated subsystems are strongly normal, but more concerning, that the intersection of two strongly normal subsystems need not be strongly normal.

Chapter 5

Simple Fusion Systems

Here, a simple fusion system is a saturated fusion system in which there are no normal, saturated subsystems.

Theorem 5.1 Let $\mathcal{F}$ be a simple fusion system on a p-group P, and suppose that $\mathcal{F}$ is realized by a finite group G. Suppose that $\mathrm{O}_{p'}(G) = 1$ and that $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G containing P. Then G is simple.

Proof: Suppose that $\mathrm{O}_{p'}(G) = 1$ and that $\mathcal{F}_P(H) \neq \mathcal{F}_P(G)$ for any proper subgroup $P \leqslant H < G$. Suppose that N is a normal subgroup of G; then $\mathcal{F}_{P \cap N}(N)$ is an normal subsystem, and since $\mathcal{F}$ is simple and $\mathrm{O}_{p'}(G) = 1$, we have that $N = G$ and G is simple.

If G is a group of minimal order with $\mathcal{F}_P(G) = \mathcal{F}$, then clearly $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup $H < G$. Also, $\mathcal{F}_P(G) = \mathcal{F}_P(G/\mathrm{O}_{p'}(G)$, and so the conditions of the first part of the theorem hold. □

Thus if a simple fusion system comes from a finite group, it comes from a simple group. The converse is not true; for example, we will prove that the only simple fusion systems on abelian p-groups are the systems $\mathcal{F}_P(P)$, where P is cyclic group of prime order. Since there are plenty of simple groups with abelian, but not prime cyclic, Sylow p-subgroups, not all fusion systems on simple groups are simple.

In order to develop a nice condition for simplicity of a fusion system, we need to examine normal subsystems on the same group.

Lemma 5.2 Let $\mathcal{F}$ be a saturated fusion system on the finite p-group P, and suppose that $\mathcal{F}'$ is a normal subsystem on P. Then for every subgroup Q of P, the index of $\mathrm{Aut}_{\mathcal{F}'}(Q)$ in $\mathrm{Aut}_{\mathcal{F}}(Q)$ is prime to p.

Proof: Let Q be any subgroup of P, and let R be a fully normalized subgroup $\mathcal{F}$-isomorphic to Q via an isomorphism ϕ. Since R is fully normalized, $\mathrm{Aut}_P(R)$ is a Sylow p-subgroup of both $\mathrm{Aut}_{\mathcal{F}'}(R)$ and $\mathrm{Aut}_{\mathcal{F}}(R)$ (Proposition 3.5), confirming the result for fully normalized subgroups. As

$$\phi^{-1}\,\mathrm{Aut}_P(Q)\phi \leqslant \mathrm{Aut}_{\mathcal{F}'}(R),$$

and since $\mathcal{F}'$ is normal in $\mathcal{F}$, we see that

$$\phi\,\mathrm{Aut}_P(R)\phi^{-1} \leqslant \mathrm{Aut}_{\mathcal{F}'}(Q),$$

and this is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$. Hence the index $|\,\mathrm{Aut}_{\mathcal{F}}(Q) : \mathrm{Aut}_{\mathcal{F}'}(Q)|$ is prime to p. □

Proposition 5.3 (Oliver) Suppose that $\mathcal{F}$ is a saturated fusion system on a finite p-group P, and suppose that $\mathcal{F}'$ is a saturated normal subsystem on P itself. If $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}'}(P)$ then $\mathcal{F}$ and $\mathcal{F}'$ coincide.

Proof: Let Q be a subgroup of P of maximal order subject to $\mathrm{Aut}_{\mathcal{F}}(P) \neq \mathrm{Aut}_{\mathcal{F}'}(P)$: by hypothesis, $Q \neq P$. Firstly assume that Q is fully normalized. Then $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ by Proposition 3.5. Since $\mathcal{F}'$ is normal in $\mathcal{F}$, $\mathrm{Aut}_{\mathcal{F}'}(Q)$ is a normal subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$ containing a Sylow p-subgroup, and so we may apply the Frattini argument. Therefore

$$\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}\,\mathrm{Aut}_{\mathcal{F}'}(Q).$$

Every automorphism of Q in $\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q))$ extends to an automorphism of $\mathrm{N}_P(Q)$ in $\mathcal{F}$, because if $\phi \in \mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q))$ then $N_\phi = \mathrm{N}_P(Q)$, and Q is fully normalized so that ϕ extends. Since $\mathrm{N}_P(Q) > Q$, it must be true that this extended automorphism also lies in $\mathcal{F}'$. Thus

$$\mathrm{N}_{\mathrm{Aut}_{\mathcal{F}}(Q)}(\mathrm{Aut}_P(Q)) \leqslant \mathrm{Aut}_{\mathcal{F}'}(Q),$$

and so therefore $\mathrm{Aut}_{\mathcal{F}}(Q) = \mathrm{Aut}_{\mathcal{F}'}(Q)$. If all automorphism groups coincide then the fusion systems coincide, by Alperin's fusion theorem. It remains to remove the hypothesis that Q is fully normalized. Let R be a subgroup of maximal order subject to $\mathrm{Aut}_{\mathcal{F}}(P) \neq \mathrm{Aut}_{\mathcal{F}'}(P)$. A similar argument to Lemma 5.2 proves the general case, as required. □

With this, we can now get this condition on simplicity.

Corollary 5.4 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Assume that $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group, and that P has no proper, non-trivial, strongly $\mathcal{F}$-closed subgroup. Then $\mathcal{F}$ is simple.

Proof: Since $\mathcal{F}$ has no strongly $\mathcal{F}$-closed subgroups, any normal subsystem $\mathcal{F}'$ must be on P, and by Proposition 5.3, since $\mathrm{Aut}_P(P) = \mathrm{Aut}_{\mathcal{F}'}(P) = \mathrm{Aut}_{\mathcal{F}}(P)$, we see that $\mathcal{F} = \mathcal{F}'$. Hence $\mathcal{F}$ is simple, as required. $\square$

In particular, we have a more restrictive corollary, which is still enough for a lot of purposes.

Corollary 5.5 Let $\mathcal{F}$ be a fusion system on a finite p-group P, which is generated by its elements of order p. Suppose that $\mathrm{Aut}_{\mathcal{F}}(P)$ is a p-group (in particular, if $\mathrm{Aut}(P)$ is a p-group) and that all elements of order p are $\mathcal{F}$-conjugate. Then $\mathcal{F}$ is simple.

Proof: Let Q be a strongly $\mathcal{F}$-closed subgroup of P. If $Q \neq 1$, then Q contains an element of order p, whence Q contains all elements of order p. Thus $Q = P$, and Corollary 5.4 proves that $\mathcal{F}$ is simple. $\square$

If P is an abelian p-group, then a fusion system $\mathcal{F}$ on P is simple only when P is cyclic of order p and $\mathcal{F} = \mathcal{F}_P(P)$. We will see this now.

Lemma 5.6 Let P be a p-group. Then $\mathcal{F}_P(P)$ is simple if and only if P is cyclic of order p.

Proof: Firstly, if $P = C_p$ then $\mathcal{F}_P(P)$ because it has no non-trivial subsystems. The other direction is only slightly harder: let Z be a central subgroup. Then $\mathcal{F} = \mathrm{C}_{\mathcal{F}}(Z)$, since every conjugation map $\theta_g : Q \to R$ in P can be extended to an automorphism of P acting centrally on Z. Therefore $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Z)$ in particular, and so $\mathcal{F}_Z(Z)$ is normal in $\mathcal{F}$. Therefore $P = C_p$, as claimed. $\square$

Proposition 5.7 Let P be an abelian p-group, and let $\mathcal{F}$ be a simple fusion system on P. Then $\mathcal{F} = \mathcal{F}_P(P)$ and $P = C_p$.

Proof: Since P is abelian, we have that $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(P)$ by Proposition 3.11, and so $\mathcal{F}_P(P)$ is an normal subsystem of $\mathcal{F}$. Hence $\mathcal{F} = \mathcal{F}_P(P)$, and by Lemma 5.6 P is cyclic of order p, as claimed. $\square$

Using this, we see that the fusion system at the prime 2 for the Janko group J_1 is not simple, but in fact we have the following, an as-yet unpublished result of Aschbacher.

Theorem 5.8 (Aschbacher) Let G be a sporadic simple group, and suppose that P is a Sylow 2-subgroup of G. Then $\mathcal{F}_P(G)$ is simple if G is not J_1.

Chapter 6

Centric Linking Systems

The previous chapters have been of an algebraic flavour. However, there is much topology involved in the theory of fusion systems, and we will see some of this now. The main difficulty is that there is a considerable amount of topological machinery involved in this, and as we are mainly concerned with fusion systems themselves, we will not be able to prove everything that we see here. (Another reason for this is that the proofs are themselves far too long and complex for a text of this type.) We will give proofs of some of the results, and give references for those we do not prove, but there is a certain amount of topology that is needed to even express some of these results.

We begin by describing the (geometric realization of the) nerve of a category, and then consider the p-completion of a topological space. Due to the complicated nature of this concept, we cannot give a precise definition of it, but we we make some, hopefully helpful, remarks about it. We move on to define the centric linking system of a finite group, which can be thought of as 'covering' a fusion system. Given an abstract fusion system, not necessarily realized by that of a group, we need a different definition of a centric linking system, and this we provide in the following section.

6.1 The Nerve of a Category

Let $\mathscr{C}$ be a small category. The nerve of a category is a simplicial complex constructed out of the morphisms, commutative triangles, and so on, of a category. In some sense it is a geometric realization of the relationships inside the category. It will be denoted by $|\mathscr{C}|$.

At this point, we should mention that technically we are building the geometric realization of the nerve, not the nerve itself. The distinction is subtle: the nerve is a simplicial set, and its geometric realization is a simplicial complex. (A simplicial set is an abstract version of a simplicial complex, which has a collection of n-simplices for all n, and maps from n-simplices into $(n+1)$-simplices – the degeneracy map – telling you which $(n+1)$-simplices are really n-simplices in disguise, and face maps the other way around, which are meant to delineate the n-simplices that form the boundary of the $(n+1)$-simplex.) For our purposes, we need not ever consider the simplicial set itself, and move straight on to its geometric realization, a simplicial complex.

To understand the complex, we must define the n-simplices. For $n \leqslant 2$, these are very easy to understand, and this intuition may be used to get a feel for the higher-dimensional simplices.

In the case where $n = 0$, the n-simplices are simply the objects of $\mathscr{C}$. For $n = 1$, the n-simplices are all non-identity morphisms $c_0 \to c_1$. (Here c_0 may be equal to c_1, as long as the map is not the identity.) The 2-simplices are all commutative triangles

$$\begin{tikzcd}
& c_1 \arrow[dr, "\psi"] & \\
c_0 \arrow[ur, "\phi"] \arrow[rr, "\phi\circ\psi"'] & & c_2
\end{tikzcd}$$

corresponding to compositions of maps $c_0 \to c_1 \to c_2$. The condition we need is that each of the two lines we used to define the triangle is really a line, so that this is really a triangle; i.e., neither of the maps ϕ nor ψ is the identity. If one were, then this is a line in disguise, and we ignore those 'faux-triangles'. Note that the map $\phi \circ \psi$ can be the identity. In this case, the triangle is pinched, in the sense that the third line is identified to a point. Thus the triangle here looks a bit like a shield, in the sense that it is a loop with two vertices, filled in.

The 3-simplices are all commutative tetrahedra; in the sequence formulation, we are looking at sequences

$$c_0 \xrightarrow{\phi} c_1 \xrightarrow{\psi} c_2 \xrightarrow{\theta} c_3 \ ,$$

where none of the maps ϕ, ψ, and θ, is the identity. One way of seeing this condition is to require that inside this tetrahedron there is a sequence of non-identity maps (1-simplices) lying in it. It is possible that some composition of the maps is the identity though, and in this case the simplex isn't really a tetrahedron in the geometric sense, but some of the lines or faces collapse. In the most extreme case all of the c_i are the same.

If the 3-simplex is as above, the 2-simplices are

$$c_0 \to c_1 \to c_2, \quad c_0 \to c_1 \to c_3, \quad c_0 \to c_2 \to c_3, \quad \text{and} \quad c_1 \to c_2 \to c_3,$$

with the obvious maps. (Some of these might not actually be 2-simplices, because the composition of two maps might be the identity, but two first and last simplices definitely exist, at least.)

The condition that none of the maps is the identity is a condition that may be imposed on higher-length sequences, and has the added advantage that it describes the $(n-1)$-simplices that form the boundary (even though some of them may be degenerate).

The nerve $|\mathscr{C}|$ of $\mathscr{C}$ is the simplicial complex got by iterating this procedure for all n. In general, this process normally will not terminate, but in certain circumstances it does.

Example 6.1 Let $\mathscr{C}_n$ be the set $\{0,\ldots,n\}$, together with a single map $i \to j$ if $i \leqslant j$, and no maps otherwise. Then $|\mathscr{C}_2|$ is simply a triangle, and in general, $|\mathscr{C}_n|$ is the n-simplex. (To see this, note that there is a unique n-simplex, and that the number of i-simplices is consistent with them all being contained within one n-simplex.)

Along with categories come functors and natural transformations.

Proposition 6.2 Let $\mathscr{C}$ and $\mathscr{D}$ be (small) categories, and let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then F induces a continuous map, $|F|$, between $|\mathscr{C}|$ and $|\mathscr{D}|$.

Proof: Since any simplicial map is continuous, we need to show that F induces a simplicial map. Certainly F induces a map from vertices to vertices, and from i-simplices to i-simplices. Furthermore, because of the functorial properties of F, we see that the boundaries of an i-simplex in $|\mathscr{C}|$ are mapped to boundaries of the image of the i-simplex in $||$, so that $|F|$ is a simplicial map, as needed. $\square$

Proposition 6.3 Let $\mathscr{C}$ and $\mathscr{D}$ be small categories, and let F and F' be two functors $\mathscr{C} \to \mathscr{D}$. Suppose that there is a natural transformation relating F and F'. Then the continuous maps $|F|$ and $|F'|$ are homotopic.

Proof: Firstly, we note that a natural tranformation $\phi : F \to F'$ induces a functor $H : \mathscr{C} \times \mathscr{C}_1 \to \mathscr{D}$, where $\mathscr{C}_1$ is the finite set described in Example 6.1. Firstly, the category $\mathscr{C}' = \mathscr{C} \times \mathscr{C}_1$ is the set of all ordered pairs (x, i), where $x \in \mathscr{C}$ and $i \in \{0,1\}$, together with morphisms (f, g), where f is a morphism in $\mathscr{C}$ and g is a morphism in $\mathscr{C}_1$.

The second step is to notice that $|\mathscr{C} \times \mathscr{C}_1|$ is isomorphic with $|\mathscr{C}| \times |\mathscr{C}_1|$, which is isomorphic to $|\mathscr{C}| \times [0,1]$.

Now everything is clear: since the functor H induces a continuous map $|H| : |\mathscr{C}| \times [0,1] \to |\mathscr{D}|$ by Proposition 6.2, we see that H induces a homotopy between the continuous maps $|\mathscr{C}| \to |\mathscr{D}|$ evaluated at 0 and $|\mathscr{C}| \to |\mathscr{D}|$ evaluated at 1; i.e., between $|F|$ and $|F'|$, as claimed. $\square$

If $F : \mathscr{C} \to \mathscr{D}$ is an equivalence of categories, then $|F| : |\mathscr{C}| \to |\mathscr{D}|$ is a homotopy equivalence.

We end this section by making the important point, that if $\mathscr{C}$ has an initial or terminal object, then $|\mathscr{C}|$ is contractible. Essentially, one may contract each simplex that originates at the (for example initial) object, to get the desired contraction mapping.

6.2 Classifying Spaces

Let G be a finite group. (It is possible to let G be a discrete group, but we will not need this generality here.) There are two ways to cast G as a category.

The first, which we write as $\mathscr{B}(G)$, is a category with one object, o_G. The set of all homomorphisms $\mathrm{Hom}(o_G, o_G)$ is the set of elements of G, with the multiplication being the usual multiplication in the group. (Since $\mathrm{Hom}(o_G, o_G)$ is a group, this definition makes sense.) The second, which we write $\mathscr{E}(G)$, is a category with objects the elements of G, and for each pair of elements there is a unique morphism between them. (Therefore this is simply the complete directed graph, with one loop on each vertex, and so for any two groups G and H with the same order we have that $\mathscr{E}(G)$ and $\mathscr{E}(H)$ are isomorphic.) Notice that G acts on $\mathscr{E}(G)$, by multiplication, and $\mathscr{E}(G)/G \cong \mathscr{B}(G)$.

Proposition 6.4 Let G and H be finite groups, and let $\phi : G \to H$ be a homomorphism. Then ϕ induces natural functors $F_\phi : \mathscr{B}(G) \to \mathscr{B}(H)$ and $F'_\phi : \mathscr{E}(G) \to \mathscr{E}(H)$.

Proof: This is clear: define $F'_\phi(g)$ to be $g\phi$, and send the unique morphism $g \to h$ to the unique morphism $g\phi \to h\phi$. The fact that ϕ is a homomorphism implies that F'_ϕ is a functor. To get F_ϕ, we perform a similar action: define $F_\phi(o_G) = o_H$, and $F_\phi(g) = g\phi$. The fact that ϕ is a homomorphism gives that F_ϕ is a functor. □

Write $BG = |\mathscr{B}(G)|$, and $EG = |\mathscr{E}(G)|$. Then, since $\mathscr{E}(G)/G \cong \mathscr{B}(G)$, we see that $EG/G \cong BG$. The topological space BG is an example of a classifying space.

Definition 6.5 Let X be a topological space. Then X is said to be a *classifying space* for a finite group G if

(i) $\pi_1(X) = G$; and

(ii) the universal covering space $\tilde{X}$ is contractible.

Since $\pi_1(BG)$ is the group G (all loops in $\mathscr{B}(G)$ look like group elements of G), and EG is contractible (as $\mathscr{E}(G)$ contains an initial object) the space BG is a classifying space for the finite group G, so that a classifying space exists for all finite groups. In fact, it is essentially unique, although we won't see this here.

Theorem 6.6 Let G be a finite group. Then, up to homotopy, there is a unique classifying space for G.

With regards fusion systems, it turns out that the classifying space is not quite the right structure to study, and we need to take the p-completion of it. This concept is very difficult to describe, even with topology, and so we will avoid it, and simply state the results that we need about it. The p-completion functor, denoted by $(-)^\wedge_p$, is a functor from the category of spaces to itself, and includes a natural transformation $\lambda : \mathrm{id} \to (-)^\wedge_p$.

A space X is called *p-complete* if $\lambda_X : X \to X^\wedge_p$ is a homotopy equivalence. The point is that the p-completion functor either immediately produces a p-complete space or never does, in the sense that repeated application of the functor fails to reach a p-complete space. If $X^\wedge_p$ is p-complete, we say that X is *p-good*, and otherwise we say that X is *p-bad*. The p-bad spaces are very bad, and so we aren't concerned with them. Spaces whose fundamental group is finite are p-good, and so BG is p-good for any finite group G. Thus it might be useful to consider $BG^\wedge_p$ in addition to BG.

An important point to make is that BG and $BG^\wedge_p$ share the same mod-p cohomology. Moreover, a map $f : X \to Y$ induces a homotopy equivalence $f^\wedge_p : X^\wedge_p \to Y^\wedge_p$ if and only if f induces an isomorphism of the cohomology rings of X and Y. Also, if X and Y are two p-good spaces, then their p-completions are homotopy equivalent if and only if there is some third space Z, and maps $X \to Z$ and $Y \to Z$ that are both mod-p homology equivalences.

6.3 The Centric Linking Systems of Groups

Arguably, the fusion system of a finite group is not quite the right object to consider from a topological point of view. Recall that if P is a Sylow p-subgroup of G, then a subgroup Q of P is $\mathcal{F}_P(G)$-centric if

$$\mathrm{C}_G(P) = \mathrm{Z}(P) \times \mathrm{O}_{p'}(\mathrm{C}_G(P)).$$

We will use the notation of, for example, [9], and denote $\mathrm{O}'_p(\mathrm{C}_G(P))$ by $\mathrm{C}'_G(P)$.

If Q and R are subgroups of P, define the *transporter* between Q and R, denoted by $\mathrm{N}_G(Q,R)$, to be

$$\mathrm{N}_G(Q,R) = \{x \in G \mid x^{-1}Qx \leqslant R\}.$$

It is easy to see that two elements g and h of $\mathrm{N}_G(Q,R)$ define the same element of $\mathcal{F}_P(G)$ if and only if $gh^{-1} \in \mathrm{C}_G(Q)$. Also, if $g \in \mathrm{N}_G(Q,R)$, then for all $x \in \mathrm{C}_G(Q)$, we see that $xg \in \mathrm{N}_G(Q,R)$. It makes sense then to 'collapse $\mathrm{N}_G(Q,R)$ on the left' by $\mathrm{C}_G(Q)$. We write

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(Q,R) = \mathrm{N}_G(Q,R)/\,\mathrm{C}_G(Q),$$

even though there is no formal quotient group to speak of. (Indeed, this is very much closer to the topological notion of quotienting, which is where one formally identifies various points.)

In the case where Q is a centric subgroup, we not only have $\mathrm{C}_G(Q)$ to quotient by, but also this other natural subgroup, $\mathrm{C}'_G(Q)$. We could also, of course, not identify morphisms at all.

Definition 6.7 Let G be a finite group and let P be a Sylow p-subgroup. The *centric linking system*, $\mathcal{L}^c_P(G)$, is a category, whose objects are all $\mathcal{F}_P(G)$-centric subgroups of P, and whose morphism sets are given by

$$\mathrm{Hom}_{\mathcal{L}^c_P(G)}(Q,R)=\mathrm{N}_G(Q,R)/\,\mathrm{C}'_G(Q).$$

The *transporter system*, $\tilde{\mathcal{L}}^c_P(G)$ is the category, ahose objects are all $\mathcal{F}_P(G)$-centric subgroups of P, and whose morphism sets are given by

$$\mathrm{Hom}_{\mathcal{L}^c_P(G)}(Q,R)=\mathrm{N}_G(Q,R).$$

In the case of $\mathcal{F}_P(G)$, we are identifying elements that factor through any element of $\mathrm{C}_G(Q)$, and in the case of $\mathcal{L}^c_P(G)$, we are identifying elements that factor through a p'-element of $\mathrm{C}_G(Q)$. Thus the same homomorphism $Q \to R$ will be labelled by different elements of G, much like in $\mathscr{B}(G)$, where the identity morphism $o_G \to o_G$ was labelled by all elements of the group.

We said that the centric linking system is the 'right' object to study from a topological point of view, but then we said that about $BG^\wedge_p$. We will back up those claims now. In fact, we show that BG and $|\mathcal{L}^c_P(G)|$ have the same p-completion.

Proposition 6.8 Let G be a finite group and let P be a Sylow p-subgroup of G. Let $\tilde{\alpha}$ denote the map $\tilde{\mathcal{L}}^c_P(G) \to \mathscr{B}(G)$ given by sending each object to o_G and each morphism to the corresponding morphism in $\mathscr{B}(G)$, so that a morphism and its image are labelled by the same element. Then the induced continuous map $|\tilde{\alpha}| : |\tilde{\mathcal{L}}^c_P(G)| \to BG$ is an $\mathbb{F}_p$-homology equivalence, and consequently,

$$|\tilde{\mathcal{L}}^c_P(G)|^\wedge_p \xrightarrow{\sim} BG^\wedge_p.$$

Similarly, there is a mod-p homology equivalence in the direction that $|\tilde{\mathcal{L}}^c_P(G)|$ and $\mathcal{L}^c_P(G)|$ are mod-p homology equivalen.

Theorem 6.9 (Broto, Levi, Oliver, [8, Proposition 1.1]) Let G be a finite group and let P be a Sylow p-subgroup of G. Then

$$BG^\wedge_p \xrightarrow{\sim} |\mathcal{L}^c_P(G)|^\wedge_p.$$

Proof: We have the maps

$$|\mathcal{L}_P^c(G)| \to |\tilde{\mathcal{L}}_P^c(G)| \leftarrow BG,$$

and so $BG_p^\wedge \xrightarrow{\sim} |\mathcal{L}_P^c(G)|_p^\wedge$, as claimed. □

6.4 Centric Linking Systems for Fusion Systems

In the previous section we defined a centric linking system for a group fusion system. It seems to act like a bridge, between the fusion system on the one hand, and the classifying space on the other, encoding both aspects of the situation.

However, in general a fusion system need not come from a finite group, and so it would be nice to have centric linking systems for arbitrary fusion systems.

Definition 6.10 Let $\mathcal{F}$ be a fusion system on the finite p-group P. A *centric linking system* associated to $\mathcal{F}$ is a category $\mathcal{L}$, whose objects are all $\mathcal{F}$-centric subgroups of P, together with a functor $\pi : \mathcal{L} \to \mathcal{F}^c$, and monomorphisms $\delta_Q : Q \to \mathrm{Aut}_{\mathcal{L}}(Q)$ for each $\mathcal{F}$-centric subgroup $Q \leqslant P$, which satisfies the following conditions:

(i) the functor π is the identity on objects, and for $Q, R \in \mathcal{F}^c$, we have that $\mathrm{Z}(Q)$ acts freely on $\mathrm{Hom}_{\mathcal{L}}(Q, R)$ by composition (identify $\mathrm{Z}(Q)$ with $\delta_Q(\mathrm{Z}(Q))$), and pi induces a bijection

$$\mathrm{Hom}_{\mathcal{L}}(P, Q)/\mathrm{Z}(Q) \cong \mathrm{Hom}_{\mathcal{F}}(Q, R);$$

(ii) for each $\mathcal{F}$-centric subgroup $Q \leqslant P$ and each $x \in P$, we have that

$$\pi : \delta_Q(x) \to \theta_x \in \mathrm{Aut}_{\mathcal{F}}(Q);\ \text{and}$$

(iii) for every $\phi \in \mathrm{Hom}_{\mathcal{L}}(Q, R)$ and $x \in Q$, we have that

$$\delta_Q(x) \circ \phi = \phi \circ \delta_R\left(\pi(\phi)(x)\right).$$

This definition might appear at first blush to be unsatisfying, for several reasons: firstly, it is not clear that centric linking systems exist; secondly, it is not clear whether they are unique even if they do exist; and the third axiom in particular appears unmotivated.

The first and second reasons are well-founded, and indeed it is not known whether centric linking systems always exist, and if they do, whether they are unique. There are positive results in this area, mainly for small-rank p-groups, which we will describe later in Theorem 6.15, but in general they remain open at this point.

We will try to motivate the definition, however, in the hopes of undermining the third reason. The first axiom simply requires that the morphism sets in $\mathcal{L}$ should behave in the same way as those for $\mathcal{L}^c_P(G)$; in that case we had that

$$\operatorname{Hom}_{\mathcal{L}}(Q,R)/\operatorname{Z}(Q) \cong \operatorname{Hom}_{\mathcal{F}}(Q,R),$$

and $\operatorname{Z}(Q)$ acts freely on the maps by composition, and so it seems natural to require this in general. The second axiom is equally important, since it makes sure that the map δ was chosen to match up with the map π, in that the automorphism of Q given by $\delta(x)$ is the same as the automorphism given by θ_x. The third axiom is more complicated: it is essentially there because proofs demand it to be. It corresponds to the commutativity of the diagram

$$\begin{tikzcd}
Q \arrow[r, "\phi"] \arrow[d, "\delta_Q(x)"'] & R \arrow[d, "\delta_R(\pi(\phi)(x))"] \\
Q \arrow[r, "\phi"'] & R
\end{tikzcd}$$

and ensures that δ interacts correctly with the morphisms.

Proposition 6.11 Let $\mathcal{F} = \mathcal{F}_P(G)$ be a fusion system on the finite group G with Sylow p-subgroup P, and let $\mathcal{L} = \mathcal{L}^c_P(G)$ be the centric linking system. Then $\mathcal{L}$ is an associated centric linking system to $\mathcal{F}$ in the sense of Definition 6.10.

Proof: Let $\mathcal{F}$ and $\mathcal{L}$ be as above. Certainly $\mathcal{L}$ is defined on the right objects, and the functor

$$\pi : \mathcal{L} \to \mathcal{F}^c$$

is just the map sending objects to the same objects, and with maps on morphism sets given by

$$\pi : \operatorname{N}_G(Q,R)/\operatorname{C}'_G(Q) \to \operatorname{N}_G(Q,R)/\operatorname{C}_G(Q).$$

The distinguished morphisms $\delta_Q : Q \to \operatorname{Aut}_{\mathcal{L}}(Q)$ are given by sending $g \in Q$ to the element $\operatorname{C}'_G(Q)g$ in $\operatorname{N}_G(Q)/\operatorname{C}'_G(Q)$ (this is a genuine quotient). We need to check the conditions.

The subgroup $\delta_Q(\operatorname{Z}(Q))$ of $\operatorname{Aut}_{\mathcal{L}}(Q)$ does indeed act freely by composition, and

$$\pi : \operatorname{Hom}_{\mathcal{L}}(Q,R)/\operatorname{Z}(Q) \to \operatorname{Hom}_{\mathcal{F}}(Q,R)$$

is definitely a bijection. Thus the first condition is satisfied. Also, by construction of δ_Q, it sends $\delta_Q(g)$ to $\theta_g \in \operatorname{Aut}_{\mathcal{F}}(Q)$, and so the second condition is satisfied.

The third condition is slightly more complicated. Suppose that ϕ arises from $g \in \operatorname{N}_G(Q,R)$, and let $x \in Q$. Here, we have that

$$\phi \circ \delta_R(\pi(\phi)(x)) = \phi \circ \delta_R(x\theta_g) = \phi \circ \delta_R(g^{-1}xg).$$

Now, $\delta_R(g^{-1}xg) = \mathrm{C}'_G(R)(g^{-1}xg) = g^{-1}\,\mathrm{C}'_G(Q)xg$, so that this expression becomes $\mathrm{C}'_G(Q)g \circ g^{-1}\,\mathrm{C}'_G(Q)xg = \mathrm{C}'_G(Q)xg$. Also,

$$\delta_Q(x) \circ \phi = \mathrm{C}'_G(Q)x\,\mathrm{C}'_G(Q)g = \mathrm{C}'_G(Q)xg,$$

and so the square commutes, as claimed. □

Having constructed many examples of associated centric linking systems, we come to a major definition in this approach to fusion systems.

Definition 6.12 A *p-local finite group* on P is a triple $(P, \mathcal{F}, \mathcal{L})$, where P is a finite p-group, $\mathcal{F}$ is a saturated fusion system over P, and $\mathcal{L}$ is an associated centric linking system of $\mathcal{F}$. If $(P, \mathcal{F}, \mathcal{L})$ is a p-local finite group, its *classifying space* is the space $|\mathcal{L}|^\wedge_p$.

In the case where $(P, \mathcal{F}, \mathcal{L})$ is a p-local finite group arising from a finite group G, then as we have seen in Theorem 6.9, we have that the classifying space of $(P, \mathcal{F}, \mathcal{L})$ is homotopy equivalent to $BG^\wedge_p$.

At this point we come to an open problem.

Question 6.13 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P.

(i) Is there a p-local finite group $(P, \mathcal{F}, \mathcal{L})$ corresponding to $\mathcal{F}$?

(ii) If $(P, \mathcal{F}, \mathcal{L})$ and $(P, \mathcal{F}, \mathcal{L}')$ are two p-local finite groups corresponding to $\mathcal{F}$, is it true that they are isomorphic in some way?

We will come to what it means for two p-local finite groups to be isomorphic soon, but we first give a little information about the questions.

In general these are open questions, but in certain cases they have been solved. In the case where $\mathcal{F} = \mathcal{F}_P(G)$ is the fusion system of a finite group, it is known that the centric linking system $\mathcal{L}^c_P(G)$ is the only centric linking system associated to $\mathcal{F}$, and thus both of the questions above have positive answers. Another example is the low-rank case: if the p-rank of the finite p-group P is strictly less than p^3, then the first question has a positive answer, and if the p-rank is strictly less than p^2, then the second question has a positive answer. The case for groups follows from the solution of the Martino–Priddy conjecture, which we will discuss later, and the low-rank case can be found in [9, Theorem E], and will also be seen later.

6.5 Obstructions to Centric Linking Systems

Like many questions about existence and uniqueness of structures, the existence and uniqueness of associated centric linking systems to a saturated fusion system $\mathcal{F}$ occurs in obstruction groups. To find these obstruction groups, we need to define an orbit category and a particular functor first.

Definition 6.14 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Then the *orbit category* $\bar{\mathcal{F}}$ of $\mathcal{F}$ is the category whose objects are the same as those of $\mathcal{F}$, and whose morphism sets are given by

$$\mathrm{Hom}_{\bar{\mathcal{F}}}(Q, R) = \mathrm{Hom}_{\mathcal{F}}(Q, R)/\operatorname{Aut}_R(R),$$

with composition of morphisms induced from that of $\mathcal{F}$.

This category is well-defined, since if ϕ and ϕ' are morphisms in $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ whose image in $\bar{\mathcal{F}}$ is the same, and ψ and ψ' are morphisms in $\mathrm{Hom}_{\mathcal{F}}(R, S)$ whose image in $\bar{\mathcal{F}}$ is the same, then $\phi\psi$ and $\phi'\psi'$ have the same image in $\bar{\mathcal{F}}$, as needed for this category to work.

Define a contravariant functor

$$\mathcal{Z}_{\mathcal{F}} : \bar{\mathcal{F}}^c \longrightarrow \mathcal{A}b,$$

by setting $\mathcal{Z}_{\mathcal{F}}(Q) = \mathrm{Z}(Q) = \mathrm{C}_P(Q)$, for each $\mathcal{F}$-centric subgroup $Q \leqslant P$. Then the obstruction for the existence of an associated centric linking system lies in the third derived functor of $\mathcal{Z}_{\mathcal{F}}$, and the obstruction to uniqueness of the associated centric linking system lies in the second derived functor of $\mathcal{Z}_{\mathcal{F}}$. The proof of this result [9, Proposition 3.1] is well beyond the scope of this lecture course. One may explicitly calculate this right derived functor, and in particular prove the following.

Theorem 6.15 (Broto, Levi, Oliver [9, Theorem E]) Let P be a finite p-group, and let $\mathcal{F}$ be a saturated fusion system over $\mathcal{F}$. If P has p-rank at most $p^3 - 1$, then there is an associated centric linking system $\mathcal{L}$ to $\mathcal{F}$, and if P has p-rank at most $p^2 - 1$, this associated centric linking system is unique.

This theorem is true because, if the p-rank of a group is at most $p^i - 1$, then the ith derived functor of $\mathcal{Z}_{\mathcal{F}}$ can be proved to vanish.

[Also, a constrained fusion system is one that contains a normal, centric subgroup.

Theorem 6.16 (BCGLO, 2005) Let $\mathcal{F}$ be a constrained saturated fusion system. Then $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G, and if G is chosen to have $\mathrm{O}_{p'}(G) = 1$ and is p-constrained (i.e., $\mathrm{C}_G(P) \leqslant \mathrm{O}_p(G)$), then G is unique.

Also, Martino–Priddy conjecture (Bob Oliver proved it.)]

Chapter 7

Glauberman Functors and Control of Fusion

In Chapter 1, we saw a few of the deeper results about fusion in finite groups, including Glauberman's ZJ-theorem. In fact, the conclusion of this theorem holds when we replace the subgroup $\mathrm{Z}(J(P))$ by certain other subgroups, the images of so-called 'Glauberman functors' [11]. A Glauberman functor is essentially a generalization of the map $P \mapsto \mathrm{Z}(J(P))$; in this chapter we will define the concept of Glauberman functors, and extend these results to arbitrary saturated fusion systems.

We also have the p-nilpotence theorems of Glauberman and Thompson, which we discussed in that chapter. These have been extended to fusion systems in general, and we will prove these extensions here. The method is to prove that a minimal counterexample is a constrained fusion system, and since they come from finite groups (Theorem 6.16), the theorem for groups proves the theorem for all saturated fusion systems.

We also include a section on control of transfer which, like control of fusion, involves finding a subgroup H of the group G such that one may detect transfer from this subgroup H. We end by considering Thompson factorization, which might or might not have an extension to saturated fusion systems. It would be a useful thing to have, if it does extend.

7.1 Glauberman Functors

We begin with the definition of the functors, first considered by Glauberman in [11].

Definition 7.1 A map W is called a *positive characteristic p-functor* if it is a map sending each finite p-group P to a characteristic subgroup $W(P)$ of P, such that

(i) $W(P) > 1$ if $P > 1$; and

(ii) If $\phi : P \to Q$ is an isomorphism of finite p-groups, then $W(P)\phi = W(Q)$.

A positive characteristic p-functor is called a *Glauberman functor* if it is a positive characteristic p-functor such that, if P is a Sylow p-subgroup of a finite group G, not involving the group $Qd(p) = (C_p \times C_p) \rtimes \mathrm{SL}_2(p)$, such that $\mathrm{C}_G(\mathrm{O}_p(G)) = \mathrm{Z}(\mathrm{O}_p(G))$, we have that $W(P) \trianglelefteqslant G$.

An example of a positive characteristic p-functor, in fact a Glauberman functor, is the map sending P to the subgroup $\mathrm{Z}(J(P))$. Other examples include the important functors K^∞ and K_∞, which we will define now.

Let P be a finite p-group, and let Q be a subgroup of P. Define $\mathcal{M}(P;Q)$ to be the set of subgroups R of P normalized by Q and such that $R/\mathrm{Z}(R)$ is abelian (i.e., R has class at most 2). The first subset, $\mathcal{M}^*(P;Q)$, is the subset of $\mathcal{M}(P;Q)$ consisting of those subgroups R for which the induced conjugation action of Q on $R/\mathrm{Z}(R)$ is trivial. The second subset, $\mathcal{M}_*(P;Q)$, consists of a collection of subgroups R in $\mathcal{M}(P;Q)$ satisfying the following condition: if $S \in \mathcal{M}(P;R)$ such that $S \leqslant Q \cap \mathrm{C}_P([\mathrm{Z}(R), S])$ and S' centralizes R, then the conjugation action of S induces the trivial action on $R/\mathrm{Z}(R)$.

Write $K_{-1}(P) = P$, and define

$$K_i(P) = \begin{cases} \langle \mathcal{M}^*(P; K_{i-1}(P)) \rangle & i \text{ odd} \\ \langle \mathcal{M}_*(P; K_{i-1}(P)) \rangle & i \text{ even} \end{cases}.$$

Definition 7.2 Let P be a finite p-group. Define

$$K^\infty(P) = \bigcap_{i \geqslant -1,\ \text{odd}} K_i(P), \text{ and } K_\infty(P) = \langle K_i(P) \mid i \geqslant 0,\ \text{even} \rangle.$$

The maps $K_\infty : P \mapsto K_\infty(P)$ and $K^\infty : P \mapsto K^\infty(P)$ are examples of positive characteristic p-functors.

Lemma 7.3 (Glauberman [11, 13.1]) Let P be a finite p-group, and let W be either of the functors K^∞ or K_∞.

(i) $W(P)$ is a characteristic subgroup of P.

(ii) $W(P)$ contains $\mathrm{Z}(P)$; in particular $W(P) > 1$ if $P \neq 1$.

(iii) If $\phi : P \to Q$ is a group isomorphism, then $W(P)\phi = W(Q)$.

Proof: If Q is a characteristic subgroup of P, then $\mathcal{M}(P;Q)$ is a collection of subgroups that is closed under any automorphism of P. If R is a subgroup in $\mathcal{M}^*(P;Q)$ an ϕ is an automorphism of P, then $R\phi \in \mathcal{M}^*(P;Q)$. Thus if Q is characteristic in P, then $\langle \mathcal{M}^*(P;Q) \rangle$

is a characteristic subgroup of P. The same is true if $R \in \mathcal{M}_*(P;Q)$, and so $\langle \mathcal{M}_*(P;Q) \rangle$ is also characteristic. Thus each of the subgroups $K_i(P)$ is characteristic, and so therefore are $K^\infty(P)$ and $K_\infty(P)$.

We notice that $\mathrm{Z}(P)$ lies $\mathcal{M}^*(P;Q)$ and $\mathcal{M}_*(P;Q)$ for all $Q \leqslant P$, since if $R = \mathrm{Z}(P)$, then there can only be the trivial action on $R/\mathrm{Z}(R)$. Thus $\mathrm{Z}(P)$ lies in $K_i(P)$ for all i, and so lies in $K_\infty(P)$ and $K^\infty(P)$.

Finally, the third part of this statement is clear, proving the lemma. □

In fact, K^∞ and K_∞ are also examples of Glauberman functors, although we will not prove this additional fact here.

7.2 The ZJ-Theorems

Glauberman's ZJ-theorem [10], stated in Chapter 1, is an important theorem about control of fusion in finite groups. It was generalized in [11] to arbitrary Glauberman functors. Here we will generalize it further.

Before we begin, recall Theorem 6.16, which states that if $\mathcal{F}$ is a fusion system on a p-group P, with an $\mathcal{F}$-centric subgroup Q, then there is a unique finite group $L = L_Q^{\mathcal{F}}$ having $\mathrm{N}_P(Q)$ as a Sylow p-subgroup, such that $\mathrm{C}_L(Q) = \mathrm{Z}(Q)$, and $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}_{\mathrm{N}_P(Q)}(L)$.

Definition 7.4 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Then $\mathcal{F}$ is said to be *$Qd(p)$-free* if $Qd(p)$ is not involved in any of the finite groups $L_Q^{\mathcal{F}}$, with $Q \in \mathcal{F}^{frc}$.

This is the definition for fusion systems corresponding to the statement that G is $Qd(p)$-free for finite groups.

Theorem 7.5 (Kessar–Linckelmann [16]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, where p is odd. Let W be a Glauberman functor. If $\mathcal{F}$ is $Qd(p)$-free, then

$$\mathcal{F} = \mathrm{N}_{\mathcal{F}}(W(P)).$$

We will sketch a proof of this now. We begin by defining two subsystems of a fusion system, that we have not needed until now.

Definition 7.6 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let Q be a subgroup of P. We denote by $Q\,\mathrm{C}_{\mathcal{F}}(Q)$ the subsystem of $\mathrm{N}_{\mathcal{F}}(Q)$ on $Q\,\mathrm{C}_P(Q)$, having as morphisms all group homomorphisms $\phi : R \to S$ for any two subgroups R and S of $Q\,\mathrm{C}_P(Q)$, which extends to a morphism $\bar{\phi} : QR \to QS$, such that $\bar{\phi}|_Q = c_x$ (for $x \in Q$).

Similarly, we denote by $\mathrm{N}_P(Q)\,\mathrm{C}_{\mathcal{F}}(Q)$ the subsystem of $\mathrm{N}_{\mathcal{F}}(Q)$ having as morphisms all group homomorphisms $\phi : R \to S$ for any two subgroups R and S of $\mathrm{N}_P(Q)$, which extends to a morphism $\bar{\phi} : QR \to QS$, such that $\bar{\phi}|_Q = c_x$ (for $x \in \mathrm{N}_P(Q)$).

A lemma of Stancu will help.

Lemma 7.7 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group. If $Q \trianglelefteqslant \mathcal{F}$, then

$$\mathcal{F} = \langle P\,\mathrm{C}_{\mathcal{F}}(Q), \mathrm{N}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))\rangle.$$

Proof: Let R be a fully normalized centric radical subgroup of P, and let ϕ be an $\mathcal{F}$-automorphism of R. We see that $Q \leqslant R$ by Proposition 3.19. Since $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$, we have that ϕ extends to an element of morphism $\bar{\phi} = \phi$ starting in $QR = R$, and so ϕ restricts to an automorphism ψ of Q. Certainly $R \leqslant N_\psi$, and it is always true that $Q\,\mathrm{C}_P(Q) \leqslant N_\psi$. Thus there is a homomorphism $\theta \in \mathrm{Hom}_{\mathcal{F}}(RQ\,\mathrm{C}_P(Q), P)$ such that $\theta|_Q = \phi|_Q$. Thus

$$\phi = \theta|_R \circ \left((\theta|_R)^{-1} \circ \phi\right).$$

The morphism $\theta|_R$ is a morphism in $\mathrm{N}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))$, and as for the first part of the factorization, this lives in $P\,\mathrm{C}_{\mathcal{F}}(Q)$. Thus $\phi \in \langle P\,\mathrm{C}_{\mathcal{F}}(Q), \mathrm{N}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))\rangle$, and by Alperin's fusion theorem we get the result. □

Proposition 7.8 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let Q be a fully normalized subgroup of P. If $\mathcal{F}$ is $Qd(p)$-free, then so are $\mathrm{N}_{\mathcal{F}}(Q)$, $\mathrm{N}_P(Q)\,\mathrm{C}_{\mathcal{F}}(Q)$, and $\mathrm{N}_{\mathcal{F}}(Q)/Q$.

The idea is to proceed by induction on the number of morphisms in a fusion system, which we denote here by $|\mathcal{F}|$. Let $\mathcal{F}$ be a minimal counterexample, so that $\mathcal{F}$ is $Qd(p)$-free, $\mathrm{N}_{\mathcal{F}}(W(P)) \neq \mathcal{F}$ where W is some Glauberman functor, but $\mathrm{N}_{\mathcal{E}}(W(Q)) = \mathcal{E}$ for any fusion system $\mathcal{E}$ such that $|\mathcal{E}| < |\mathcal{F}|$.

Step 1: $\mathrm{O}_p(\mathcal{F}) > 1$. If $\mathcal{F}$ has no non-trivial normal subgroups, then for every fully normalized subgroup Q of P, we have that $\mathrm{N}_{\mathcal{F}}(Q) < \mathcal{F}$. Now $\mathrm{N}_{\mathcal{F}}(Q)$ is $Qd(p)$-free by Proposition 7.8, and this can be used to get a contradiction.

Set $Q = \mathrm{O}_p(\mathcal{F})$ and $R = Q\,\mathrm{C}_P(Q)$. Then R is $\mathcal{F}$-centric by Lemma 3.15. If $R = Q$, then $\mathcal{F}$ would have a normal $\mathcal{F}$-centric subgroup, and so is constrained. Hence $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G, and Glauberman's ZJ-theorem gives the desired contradiction.

Step 2: $\mathcal{F} = P\,\mathrm{C}_{\mathcal{F}}(Q)$. To prove this, we can use Lemma 7.7.

From here we can complete the proof: since $\mathcal{F} = P\,\mathrm{C}_{\mathcal{F}}(Q)$ is $Qd(p)$-free, so is $\mathcal{F}/Q$ by Proposition 7.8, and so by induction

$$\mathcal{F}/Q = \mathrm{N}_{\mathcal{F}/Q}(W(P/Q)).$$

If S is the preimage of $W(P/Q)$ in R, then $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(S)$, since this is true in general. Therefore $S \leqslant Q = \mathrm{O}_p(\mathcal{F})$. Since $W(P/Q) \neq 1$, and so S contains Q properly. This contradication proves the theorem, as needed.

7.3 Fusion system p-complement theorems

Thompson's p-complement theorem is an important result in local analysis. This can be extended to all fusion systems, as done by Dí az, Glesser, Mazza, and Park.

Theorem 7.9 (Thompson) Let G be a finite group and let P be a Sylow p-subgroup of G. Suppose that p is odd or that $p = 2$ and G is S_4-free. If $\mathrm{C}_G(\mathrm{Z}(P))$ and $\mathrm{N}_G(J(P))$ are both p-nilpotent, then G is p-nilpotent.

A direct extension to fusion systems is possible.

Theorem 7.10 (DGMP) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Assume that p is odd or that $p = 2$ and $\mathcal{F}$ is S_4-free. If $\mathrm{C}_{\mathcal{F}}(\mathrm{Z}(P)) = \mathrm{N}_{\mathcal{F}}(J(S)) = \mathcal{F}_P(P)$, then $\mathcal{F} = \mathcal{F}_P(P)$.

As we saw in Chapter 1, Glauberman refined this theorem to the fact that (for p odd) if $\mathrm{N}_G(\mathrm{Z}(J(P)))$ possesses a normal p-complement, then G does. This has also been extended to fusion systems by Kessar and Linckelmann.

Theorem 7.11 Let p be an odd prime, and let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. If $\mathrm{N}_{\mathcal{F}}(\mathrm{Z}(J(P))) = \mathcal{F}_P(P)$, then $\mathcal{F} = \mathcal{F}_P(P)$.

We will sketch a proof of this now. We proceed, as with the proof of Theorem 7.5, by induction on $|\mathcal{F}|$. Assume that $\mathcal{F}$ is a minimal counterexample. If $\mathcal{E}$ is a (saturated) subsystem of $\mathcal{F}$, then $\mathcal{E}$ satisfies the condition of the theorem, and so $\mathcal{E} = \mathcal{F}_P(P)$

Step 1: $\mathrm{O}_p(\mathcal{F}) \neq 1$. Since $\mathcal{F} \neq \mathcal{F}_P(P)$, there is some fully normalized subgroup Q of P, such that $\mathrm{N}_{\mathcal{F}}(Q) \neq \mathcal{F}_{\mathrm{N}_P(Q)}(\mathrm{N}_P(Q))$, else by Alperin's fusion theorem, we have a contradiction. Choose such a fully normalized subgroup Q such that $R = \mathrm{N}_P(Q)$ is of maximal order. We claim that $R = P$.

Choose Q such that $\mathrm{Z}(J(R))$ is fully normalized. (By Proposition 3.6, there is a morphism $\psi : \mathrm{N}_P(\mathrm{Z}(J(R))) \to P$ such that the image is fully normalized. We have that

$$\mathrm{N}_P(Q) = R \leqslant \mathrm{N}_P(R) \leqslant \mathrm{N}_P(\mathrm{Z}(J(R))),$$

and since $\mathrm{N}_P(\mathrm{Z}(J(R)))\phi \leqslant \mathrm{N}_P(\mathrm{Z}(J(R))\phi)$ and $\mathrm{N}_P(\mathrm{Z}(J(R)))\phi$ is fully normalized, so is the image $Q\phi$. Replacing Q by $Q\phi$, consider the fusion system $\mathrm{N}_{\mathcal{F}}(\mathrm{Z}(J(R)))$ on $\mathrm{N}_P(\mathrm{Z}(J(R)))$.

Since $R < P$, we also have $R < \mathrm{N}_P(R)$, and so $R < \mathrm{N}_P(\mathrm{Z}(J(R)))$. By choice of Q, we have that

$$\mathrm{N}_{\mathcal{F}}(\mathrm{Z}(J(R))) = \mathcal{F}_{\mathrm{N}_P(\mathrm{Z}(J(R)))}(\mathrm{N}_P(\mathrm{Z}(J(R)))).$$

In particular, $\mathrm{N}_{\mathrm{N}_{\mathcal{F}}(Q)}(\mathrm{Z}(J(R))) = \mathcal{F}_R(R)$, and by choice of $\mathcal{F}$, we have $\mathrm{N}_{\mathcal{F}}(Q) = \mathcal{F}_R(R)$, a contradiction. Hence $R = P$, and so $Q \trianglelefteqslant P$. Since $\mathrm{N}_{\mathcal{F}}(Q) \neq \mathcal{F}_P(P)$, we see that $Q \trianglelefteqslant \mathcal{F}$, and so $\mathrm{O}_p(\mathcal{F}) \neq 1$.

Setting $Q = \mathrm{O}_p(\mathcal{F})$, we note that $Q < P$. In particular,

$$\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathrm{N}_{\mathcal{F}}(\mathrm{Z}(J(P)))} = \mathrm{Aut}_P(P).$$

Step 2: $P\,\mathrm{C}_{\mathcal{F}}(Q) = \mathcal{F}_P(P)$. If this does not hold, then $\mathcal{F} = P\,\mathrm{C}_{\mathcal{F}}(Q)$, and this leads to a contradiction.

Then by Lemma 7.7, we see that $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))$. Since Q is fully normalized, it is fully centralized, and so $Q\,\mathrm{C}_P(Q)$ is centric. Thus $\mathcal{F}$ is constrained, and so arises from a finite group. Since all finite groups satisfy the theorem, we have the desired result.

7.4 Transfer and Thompson Factorization

The Glauberman functors K^∞ and K_∞ have other nice properties, like controlling transfer. For finite groups, transfer tells us that $P/P\cap G' = G/\mathrm{O}^p(G)$, and the focal subgroup theorem tells us that

$$P \cap G' = \langle xy^{-1} | x \text{ and } y \text{ are } G\text{-conjugate}\rangle,$$

and in particular notice that the right-hand side is controlled by the fusion system.

A positive characteristic p-functor W is said to *control p-transfer in G* if

$$P \cap G' = P \cap (\mathrm{N}_G(W(P)))'.$$

Theorem 7.12 (Glauberman [11]) The Glauberman functors K^∞ and K_∞ control p-transfer in every finite group if $p \geqslant 5$.

As we mentioned, the subgroup $P \cap G'$ is determined by the fusion pattern, by the focal subgroup theorem. Thus for any fusion theorem $\mathcal{F}$ on a finite p-group P define, for $Q \leqslant P$,

$$[Q, \mathcal{F}] = \langle x^{-1}(x\phi) \mid x \in Q,\ \phi \in \mathrm{Hom}_{\mathcal{F}}(\langle x\rangle, P)\rangle.$$

The subgroup $[P, \mathcal{F}]$ will be called the *$\mathcal{F}$-focal subgroup.* If $\mathcal{F} = \mathcal{F}_P(G)$, then the $\mathcal{F}$-focal subgroup is simply the focal subgroup.

Definition 7.13 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. A positive characteristic p-functor W is said to *control transfer in* $\mathcal{F}$ if

$$[P, \mathcal{F}] = [P, \mathrm{N}_{\mathcal{F}}(W(P))].$$

The statement of the corresponding theorem for all fusion systems is now clear.

Theorem 7.14 (DGMP) The Glauberman functors K^∞ and K_∞ control transfer in every saturated fusion system on a finite p-group if $p \geqslant 5$.

Glauberman's ZJ-theorem is not as often used in the literature as Thompson factorization, although they can accomplish the same goal; for example, the soluble 2-signalizer functor theorem can be proved using the ZJ-theorem [6] or using Thompson factorization [3, Chapter 15].

Definition 7.15 Let G be a finite group, and let M be a faithful $\mathbb{F}_pG$-module. Then M is called a *failure of factorization module* if there exists a non-trivial elementary abelian p-subgroup Q such that

$$|Q| \geqslant |M/\mathrm{C}_M(Q)|.$$

We may now state the general case of Thompson factorization.

Theorem 7.16 (Thompson Factorization) Let G be a finite group with $F^*(G) = \mathrm{O}_p(G)$, and set $M = \Omega_1(\mathrm{Z}(\mathrm{O}_p(G)))$. If M is not a failure of factorization module for $G/\mathrm{C}_G(M)$, then

$$G = \mathrm{N}_G(J(P))\,\mathrm{C}_G(\Omega_1(\mathrm{Z}(P))).$$

A translation of most of the terms here into the language of fusion systems is possible, but at the moment it is not known whether this theorem can be generalized to all fusion systems. For soluble groups where $\mathrm{O}_{p'}(G) = 1$, if p is at least 5, then the conclusion of Thompson factorization always holds, and we have no need to define failure of factorization modules for fusion systems (or indeed, the generalized Fitting subsystem, but see Chapter 8). It is possible that this theorem carries over to soluble fusion systems, although there might be a problem arising from the fact that a soluble fusion system need not come from a soluble group. Perhaps extra conditions are required.

Chapter 8

The Generalized Fitting Subsystem

In [4], Aschbacher explores the concepts around quasisimple subsystems, components, the generalized Fitting subsystem, and L-balance. The definition of $\mathrm{O}^p(\mathcal{F})$ is quite complicated; even the definition of the subgroup of P on which it is a subsystem is complicated to define. It is designed as an analogue for an arbitrary subgroup of the subgroup $P \cap \mathrm{O}^p(G)$ for a finite group.

The interesting case is where $\mathcal{F} = \mathrm{O}^p(\mathcal{F})$, and this is the starting point for the definition of a quasisimple system. From here we can define components, and then the layer, $E(\mathcal{F})$, and then the generalized Fitting subsystem. From here, we prove that this subsystem contains its centralizer, as hoped, and give a version of L-balance for fusion systems, which does not require the classification of the finite simple groups, but does not imply the corresponding theorem for groups either.

8.1 Characteristic, Subnormal, and Central Subsystems

The three types of subsystem given in the title to this section are relatively easy to define, and we will start with subnormal subsystems.

Definition 8.1 Let $\mathcal{F}$ be a saturated fusion system, and let $\mathcal{E}$ be a subsystem of $\mathcal{F}$. Then $\mathcal{E}$ is said to be *subnormal* if there is a chain

$$\mathcal{E} = \mathcal{E}_0 \trianglelefteq E_1 \trianglelefteq \cdots \trianglelefteq \mathcal{E}_n = \mathcal{F},$$

of subsystems, each normal in the next. The *defect* of $\mathcal{E}$ is the smallest n ranging over all sequences of the above form. We write $\mathcal{E} \trianglelefteq\trianglelefteq \mathcal{F}$ to denote that $\mathcal{E}$ is subnormal in $\mathcal{F}$.

Since a normal subsystem is saturated, we see that a subnormal subsystem is also saturated.

Lemma 8.2 Let $\mathcal{F}$ be a saturated fusion system and let $\mathcal{E}$ and $\mathcal{E}'$ be subnormal systems of $\mathcal{F}$ of defects n and n' respectively. Then $\mathcal{E} \cap \mathcal{E}'$ is subnormal of defect at most $n + n'$.

Proof: This is clear: proceed by induction on n', noting that for n' equal to 0, the result is trivial. Let

$$\mathcal{E} = \mathcal{E}_0 \trianglelefteq \mathcal{E}_1 \trianglelefteq \cdots \trianglelefteq \mathcal{E}_n = \mathcal{F} \text{ and } \mathcal{E}' = \mathcal{E}_0' \trianglelefteq \mathcal{E}_1' \trianglelefteq \cdots \trianglelefteq \mathcal{E}_{n'}' = \mathcal{F}$$

be two subnormal series. By induction $\mathcal{E} \cap \mathcal{E}_1'$ is subnormal of defect at most $n + n' - 1$, and since $\mathcal{E} \cap \mathcal{E}' \trianglelefteq \mathcal{E} \cap \mathcal{E}_1'$, we are done. □

In order to consider characteristic subgroups, we need to know how automorphisms of a fusion system interact with notions of normality.

Lemma 8.3 Let $\mathcal{F}$ be a fusion system, and let ϕ be an automorphism of $\mathcal{F}$. Let $\mathcal{E}$ be a subsystem of $\mathcal{F}$, on a subgroup Q.

(i) If Q is weakly $\mathcal{F}$-closed then so is $Q\phi$.

(ii) If Q is strongly $\mathcal{F}$-closed then so is $Q\phi$.

(iii) If $\mathcal{E}$ is saturated then $\mathcal{E}\phi$ is saturated.

(iv) If $\mathcal{E}$ is normal or subnormal, then $\mathcal{E}\phi$ is normal or subnormal respectively.

(v) If $\mathcal{E}$ is strongly normal in $\mathcal{F}$ then $\mathcal{E}\phi$ is strongly normal in $\mathcal{F}$.

Now that we know that if $\mathcal{E} \trianglelefteq \mathcal{F}$ then $\mathcal{E}\phi \trianglelefteq \mathcal{F}$, we may define characteristic subsystems.

Definition 8.4 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{E}$ be a subsystem of $\mathcal{F}$. Then $\mathcal{E}$ is said to be *characteristic* if $\mathcal{E} \trianglelefteq \mathcal{F}$ and $\mathcal{E}\phi = \mathcal{E}$ for all $\phi \in \mathrm{Aut}(\mathcal{F})$. We denote this by $\mathcal{E} \operatorname{char} \mathcal{F}$.

If a subsystem $\mathcal{E}$ is normal then in particular the subgroup Q that it is on is strongly $\mathcal{F}$-closed, and in particular normal. Hence if $\mathcal{E}$ is, for example, the smallest normal subsystem on a particular subgroup then it is characteristic.

Proposition 8.5 Let $\mathcal{F}$ be a fusion system on a finite p-group P. Suppose that $\mathcal{E} \trianglelefteq \mathcal{E}' \trianglelefteq \mathcal{F}$, and write Q for the subgroup on which $\mathcal{E}'$ acts. Suppose that all $\mathcal{F}$-automorphisms of Q induce automorphisms on $\mathcal{E}$. Then $\mathcal{E} \trianglelefteq \mathcal{F}$. In particular, if $\mathcal{E} \operatorname{char} \mathcal{E}' \trianglelefteq \mathcal{F}$, then $\mathcal{E} \trianglelefteq \mathcal{F}$.

We already know of a characteristic subsystem, namely $\mathrm{O}_p(\mathcal{F})$; to see this, note that $Q \trianglelefteq \mathcal{F}$ if and only if $\mathcal{F}_Q(Q) \trianglelefteq \mathcal{F}$. Write R for the subgroup $\mathrm{O}_p(\mathcal{F})$, on which the subsystem $\mathcal{F}_R(R)$, which we also denote by $\mathrm{O}_p(\mathcal{F})$, acts. Then Lemma 8.3 tells us that $Q\phi \trianglelefteq \mathcal{F}$ for any automorphism ϕ of $\mathcal{F}$, and since R is the product of all normal subgroups of $\mathcal{F}$, we get the result that $\mathcal{F}_R(R) \operatorname{char} \mathcal{F}$.

In general, if Q is a strongly $\mathcal{F}$-closed subgroup, then $\mathcal{F}_Q(Q)$ is preserved under any automorphism of $\mathcal{F}$, and so we get the obvious result.

Proposition 8.6 Suppose that Q is a strongly $\mathcal{F}$-subgroup of the finite p-group P, and let $\mathcal{F}$ be a saturated fusion system on P. Then $Q \trianglelefteq \mathcal{F}$ if and only if $\mathcal{F}_Q(Q)$ is a characteristic subsystem.

Having produced one characteristic subsystem, we now consider another, namely the centre.

Definition 8.7 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Define the *centre* of $\mathcal{F}$ to be the subgroup

$$\mathrm{Z}(\mathcal{F}) = \{z \in \mathrm{O}_p(\mathcal{F}) \mid z\phi = z \text{ for all } \phi \in \mathrm{Aut}_{\mathcal{F}}(\mathrm{O}_p(\mathcal{F}))\},$$

and also write $\mathrm{Z}(\mathcal{F})$ for the system $\mathcal{F}_{\mathrm{Z}(\mathcal{F})}(\mathrm{Z}(\mathcal{F}))$.

By Proposition 8.6, the centre is characteristic if and only if $\mathrm{Z}(\mathcal{F})$ is strongly $\mathcal{F}$-closed and $\mathrm{Z}(\mathcal{F})$ is a normal subgroup of $\mathcal{F}$. The first statement is true since an morphism on any subgroup of $\mathrm{Z}(\mathcal{F})$ fixes that subgroup. The second statement is true since $\mathrm{Z}(\mathcal{F}) \leqslant \mathrm{O}_p(\mathcal{F})$.

Proposition 8.8 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $Z = \mathrm{Z}(\mathcal{F})$. Then there is a natural one-to-one map between the saturated subsystems of $\mathcal{F}/Z$, and the saturated subsystems of $\mathcal{F}$ containing $\mathcal{F}_Z(Z)$. This bijection respects normality.

8.2 Quasisimple Subsystems

Let P be a finite group, and let $\mathcal{F}$ be a saturated fusion system on P. We define the subgroup $[P, \mathrm{O}^p(\mathcal{F})]$ by

$$[P, \mathrm{O}^p(\mathcal{F})] = \langle [Q, \mathrm{O}^p(\mathrm{Aut}_{\mathcal{F}}(Q))] \mid Q \in \mathcal{F}^{fc} \rangle.$$

This is an internal definition of this subgroup, which has several other definitions, some of which are more useful in proving theorems.

Note that the $\mathcal{F}$-focal subgroup $[P, \mathcal{F}]$ contains the subgroup $[P, \mathrm{O}^p(\mathcal{F})]$ in general. It turns out that if $\mathcal{F} = \mathcal{F}_P(G)$, then

$$[P, \mathrm{O}^p(\mathcal{F})] = P \cap \mathrm{O}^p(G).$$

Since $P \cap \mathrm{O}^p(G)$ is often smaller than $P \cap G'$ (and one is equal to P if and only if the other is) the subgroup $[P, \mathrm{O}^p(\mathcal{F})]$ might well be more useful in the determination of factor groups of G.

In [4], Aschbacher defines a characteristic subsystem $\mathrm{O}^p(\mathcal{F})$ on the subgroup $[P, \mathrm{O}^p(\mathcal{F})]$. We are mainly interested in the case where $\mathrm{O}^p(\mathcal{F}) = \mathcal{F}$, and we give the definition of a quasisimple subsystem using this definition.

Definition 8.9 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group $P \neq 1$. Then $\mathcal{F}$ is said to be *quasisimple* if $\mathcal{F} = \mathrm{O}^p(\mathcal{F})$ and $\mathcal{F}/\mathrm{Z}(\mathcal{F})$ is simple.

This is the only situation in which we are interested in the subsystem $\mathrm{O}^p(\mathcal{F})$, and by the following proposition, this means that we do not need to really worry about it at all.

Proposition 8.10 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Then the following are equivalent:

(i) $\mathcal{F}$ is quasisimple;

(ii) $\mathcal{F} = \mathrm{O}^p(\mathcal{F})$ and every proper normal subsystem of $\mathcal{F}$ is contained within $\mathrm{Z}(\mathcal{F})$; and

(iii) $[P, \mathrm{O}^p(\mathcal{F})] = P$, $\mathcal{F}/\mathrm{Z}(\mathcal{F})$ is simple, and every proper normal subsystem of $\mathcal{F}$ on P.

The third equivalent condition of the proposition is a statement that is independent of the definition of $\mathrm{O}^p(\mathcal{F})$, and so may be taken as the definition of a quasisimple subsystem for our purposes.

Lemma 8.11 Let P be a finite p-group, and let $\mathcal{F}$ be a saturated fusion system on P. Assume that $P/\mathrm{Z}(\mathcal{F})$ is abelian. Then $P \trianglelefteq \mathcal{F}$ and $\mathcal{F}$ is not quasisimple.

8.3 Components and the Generalized Fitting Subsystem

A subsystem $\mathcal{C}$ of a saturated fusion system $\mathcal{F}$ will be called a *component* if $\mathscr{C}$ is quasisimple and subnormal. We write $\mathrm{Comp}(\mathcal{F})$ for the set of components of $\mathcal{F}$.

Proposition 8.12 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group, and suppose that $\mathcal{E}$ is a subnormal subsystem on a subgroup Q. Let $\mathcal{C}$ be an element of $\mathrm{Comp}(\mathcal{F})$, on a subgroup R. then either $\mathcal{C} \in \mathrm{Comp}(\mathcal{E})$ or Q and R centralize each other. In particular, if $\mathcal{C}'$ is another element of $\mathrm{Comp}(\mathcal{F})$ on a subgroup S, then $[R, S] = 1$ and $R \cap S \leqslant \mathrm{Z}(\mathcal{C}) \cap \mathrm{Z}(\mathcal{C}')$.

If $\mathcal{C}$ is a component, which is on a subgroup Q, then $\mathcal{C}$ is the only component on the subgroup Q, since if there were another one then Q is abelian, a contradiction by Lemma 8.11.

Definition 8.13 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Define the *layer* on $\mathcal{F}$ to be the subsystem generated by the elements of $\mathrm{Comp}(\mathcal{F})$. The *generalized Fitting subsystem* of $\mathcal{F}$ is defined to be the subsystem generated by $\mathrm{O}_p(\mathcal{F})$ and $E(\mathcal{F})$. It is denoted by $F^*(\mathcal{F})$.

Theorem 8.14 The subsystems $E(\mathcal{F})$ and $F^*(\mathcal{F})$ are characteristic subsystems of $\mathcal{F}$.

We also have an analogue for fusion systems to the Hall–Bender theorem on the (generalized) Fitting subgroup.

Theorem 8.15 Let $\mathcal{F}$ be a saturated fusion system. Then $\mathrm{C}_{\mathcal{F}}(F^*(\mathcal{F})) = \mathrm{Z}(F^*(\mathcal{F}))$.

8.4 Balance for Quasisimple Subsystems

An important property of the layer of a finite group is the following theorem [14], which is called $L_{p'}$-balance. If H is a subnormal subgroup of G, we say that H is a *p-component* of G if $H = \mathrm{O}^{p'}(H)$, and $H/\mathrm{O}_{p'}(H)$ is quasisimple.

Theorem 8.16 ($L_{p'}$-balanace) If G is a finite group, and P is a p-group acting on G, then

$$L_{p'}(\mathrm{C}_G(P)) \leqslant L_{p'}(G),$$

where $L_{p'}(X)$ denotes the subgroup generated by all p-components of X.

For $p \neq 2$, this theorem is only known thanks to the classification of the finite simple groups, although if a proof of the *p-Schreier property* – that for a finite simple group G and a Sylow p-subgroup P of G, the group $\mathrm{C}_{\mathrm{Aut}(G)}(P)$ is soluble – is found independent of the classification, then this requirement would be removed. The analagous result for fusion systems requires no such difficult theorems.

Theorem 8.17 (Aschbacher [4]) If $\mathcal{F}$ is a saturated fusion system on a p-group P, and Q is a fully normalized subgroup of P, then

$$E(\mathrm{N}_{\mathcal{F}}(Q)) \leqslant E(\mathcal{F}).$$

We begin by dealing with the case where a fully normalized subgroup centralizes $E(\mathcal{F})$.

Lemma 8.18 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a fully normalized subgroup. Suppose that Q centralizes $E(\mathcal{F})$. Then $E(\mathcal{F}) = E(\mathrm{N}_{\mathcal{F}}(Q))$.

Proof: Let $\mathcal{E} = \mathrm{N}_{\mathcal{F}}(Q)$, and let $\mathcal{F}' = E(\mathcal{F})$, on the subgroup R. Since Q centralizes $\mathcal{F}'$, we have that $\mathcal{F}' \leqslant \mathcal{E}$, and since $\mathcal{F}'$ is generated by components, we see that

$$E(\mathcal{F}) \leqslant E(\mathrm{N}_{\mathcal{F}}(Q)).$$

Now suppose that $\mathcal{C}$ is a component of $\mathcal{E}$. Then $\mathcal{C}$ centralizes $\mathrm{O}_p(\mathcal{F})$, and if $\mathcal{C} \nleqslant \mathcal{F}'$ then $\mathcal{C}$ centralizes $\mathcal{F}'$, so that

$$\mathcal{C} \leqslant \mathrm{C}_{\mathcal{F}}(F^*(\mathcal{F})) = \mathrm{Z}(\mathcal{F}).$$

This contradicts the statement that $\mathcal{C} = \mathrm{O}^p(\mathcal{C})$. Hence $E(\mathcal{F}) = E(\mathrm{N}_{\mathcal{F}}(Q))$, as claimed. $\square$

The proof is a sequence of reductions, starting with a minimal counterexample. So let $\mathcal{F}$ be a saturated fusion system, and let Q be a fully normalized subgroup, with a component $\mathcal{C}$ with $\mathcal{C} \nleqslant E(\mathcal{F})$. Choose $\mathcal{F}$ to be a counterexample with the underlying subgroup of minimal order. Note that by Lemma 8.18 we have that Q does not centralize $E(\mathcal{F})$.

Step 1: $\mathrm{Z}(\mathcal{F}) = 1$.

Step 2: $\mathrm{O}_p(\mathcal{F}) = 1$.

Step 3: Let R be the subgroup over which $E(\mathcal{F})$ is a subsystem. We may choose Q and $\mathcal{C}$ such that either $Q \leqslant R$ or $Q \cap R = 1$.

The rest of the proof is a technical examination of the relationship between $\mathrm{N}_{\mathcal{F}}(Q)$ and $\mathrm{N}_{E(\mathcal{F})}(Q)$.

Chapter 9

Open Problems and Conjectures

In this short chapter, we will consider some of the open problems and conjectures involved in fusion systems. The first is a general plan.

Statement 9.1 Classify all simple fusion systems on p-groups.

This might be too general a question, so here is a potentially simpler question.

Question 9.2 Are the Solomon fusion systems the only exotic, simple fusion systems in characteristic 2?

Sticking with simple fusion systems, we can ask more general questions about them. In [5], Aschbacher gives an example of a strongly $\mathcal{F}$-closed subgroup that does not possess a strongly normal subsystem on it.

Question 9.3 Is there a simple fusion system with a strongly $\mathcal{F}$-closed subgroup?

We have two definitions of normality, which are definitely not the same, in the sense that there are normal subsystems that are not strongly normal. We define a simple fusion system to be one without normal subsystems, and a *weakly simple* fusion system to be one without strongly normal subsystems.

Question 9.4 Is every weakly simple fusion system simple?

Moving off simple systems, there are other questions to consider.

Question 9.5 Does every fusion system have an associated centric linking system? Is it unique?

The Martino–Priddy conjecture, in effect, proved that if $\mathcal{F} = \mathcal{F}_P(G)$, then the associated centric linking system is unique. This includes all constrined fusion systems, and in recent work of Kessar and Linckelmann, all $Qd(p)$-free systems. In general these questions remain open.

Question 9.6 What is the right notion of a morphism of p-local finite groups?

There are some candidates for morphisms of p-local finite groups, and each of them currently has its deficiencies. In the future, some of the kinks in the topological approach to these things should be ironed out.

Finally, we consider possibly the most important question.

Question 9.7 If $\mathcal{F}$ is the fusion system of a p-block, then is there a finite group G such that $\mathcal{F} = \mathcal{F}_P(G)$?

If this is true, then for $\mathcal{F}$ a block fusion system, there is a unique p-completed classifying space associated to $\mathcal{F}$, which might well help with understanding the structure of blocks of finite groups. It is not clear how to attack this problem.

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